Hybrid models

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Hilbert XIII

Hilbert (1900): Write solution $h(x_1,\dots,x_n)$ of polynomial with coefficients $x_1,\dots,x_n$ as composition of functions u_j of 1-2 variables.

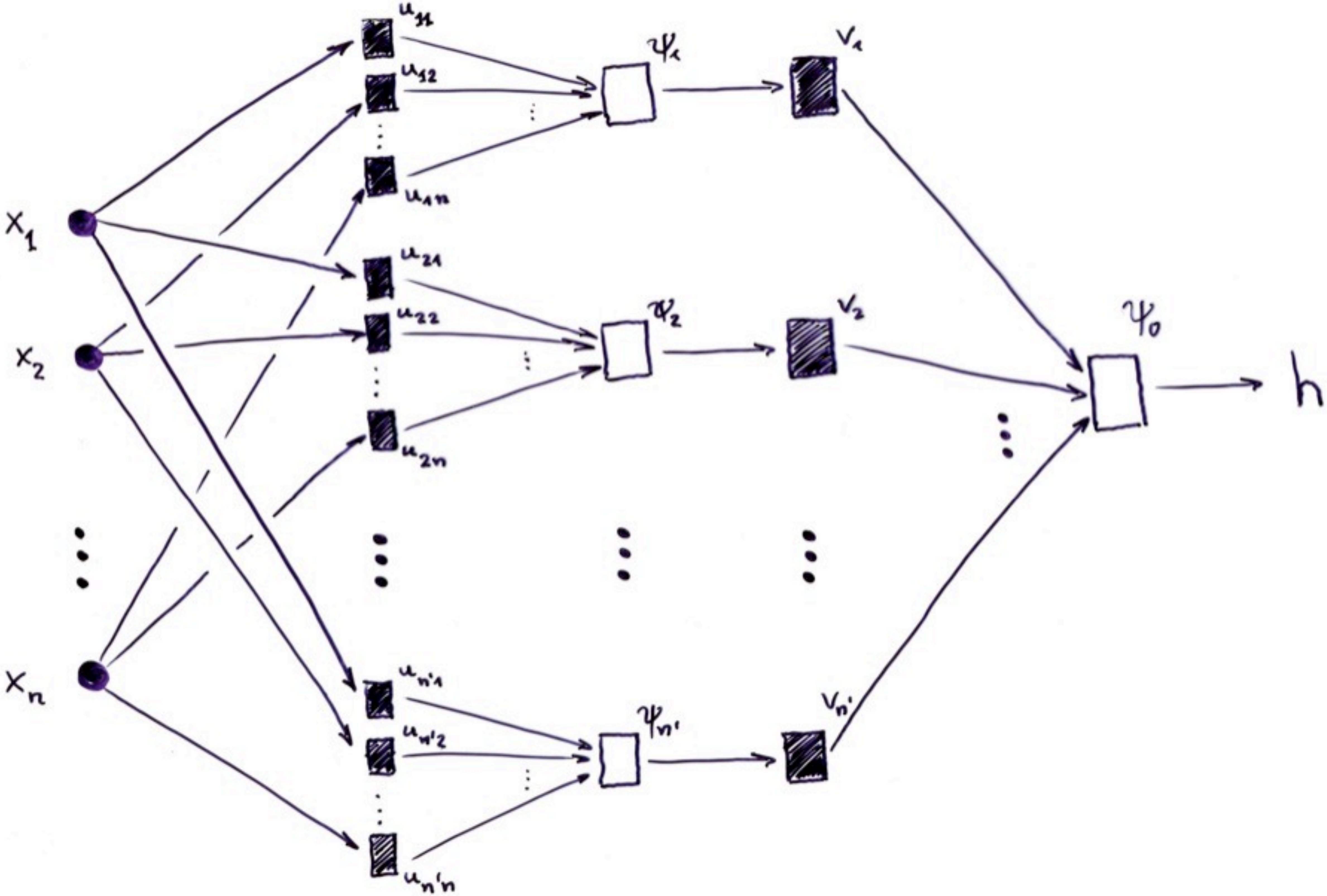
Arnol'd, Kolmogorov (1963):

YES $\quad h(x_1,\dots,x_n) = \sum_{i=1}^{2n+1} u_i\left(\sum_{j=1}^{n} u_{i,j}(x_j)\right)$ for $h, u \in C^0$

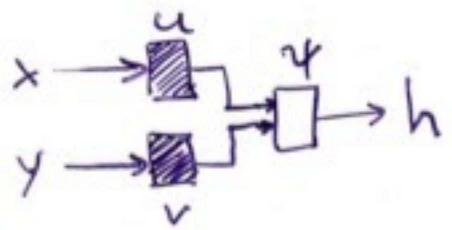
Vitushkin (1954, 1977):

NO $\quad$ for $h \in C^k(\mathbb{R}^n)$, $u_j \in C^{k'}(\mathbb{R}^{n'})$, if $\frac{k}{n} < \frac{k'}{n'}$

Example: Kolmogorov graph



Identification of $\psi(u,v) = h$



(1) $$\psi(u(x), v(y)) = h(x,y).$$

By differentiation with respect to x, y:

(2) $$\frac{\psi_{uv}}{\psi_u \psi_v}(u(x), v(y)) = \frac{h_{xy}}{h_x h_y}(x,y)$$

Assume: 1. monotonicity: $\psi_u \neq 0, \ \psi_v \neq 0$

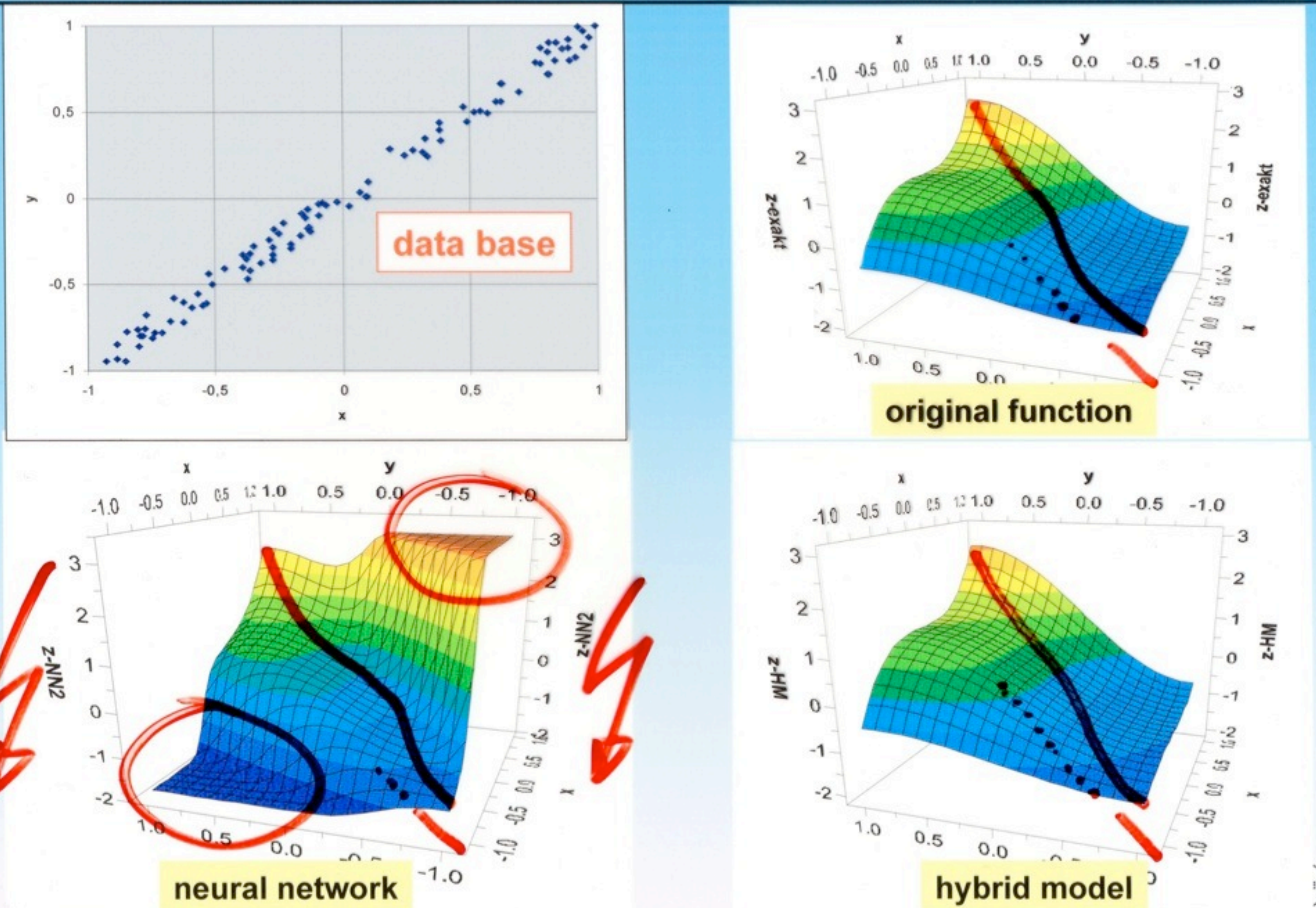
2. nondegeneracy: $\det D_{(u,v)} \begin{pmatrix} \psi \\ \psi_{uv}/(\psi_u \psi_v) \end{pmatrix} \neq 0.$

Result: Solve system (1), (2) locally uniquely for black boxes u, v.

Example: Along curves $x(s), y(s)$ find unique

$$u = u(x(s))$$
$$v = v(y(s))$$

Structured hybrid models: data base and extrapolation

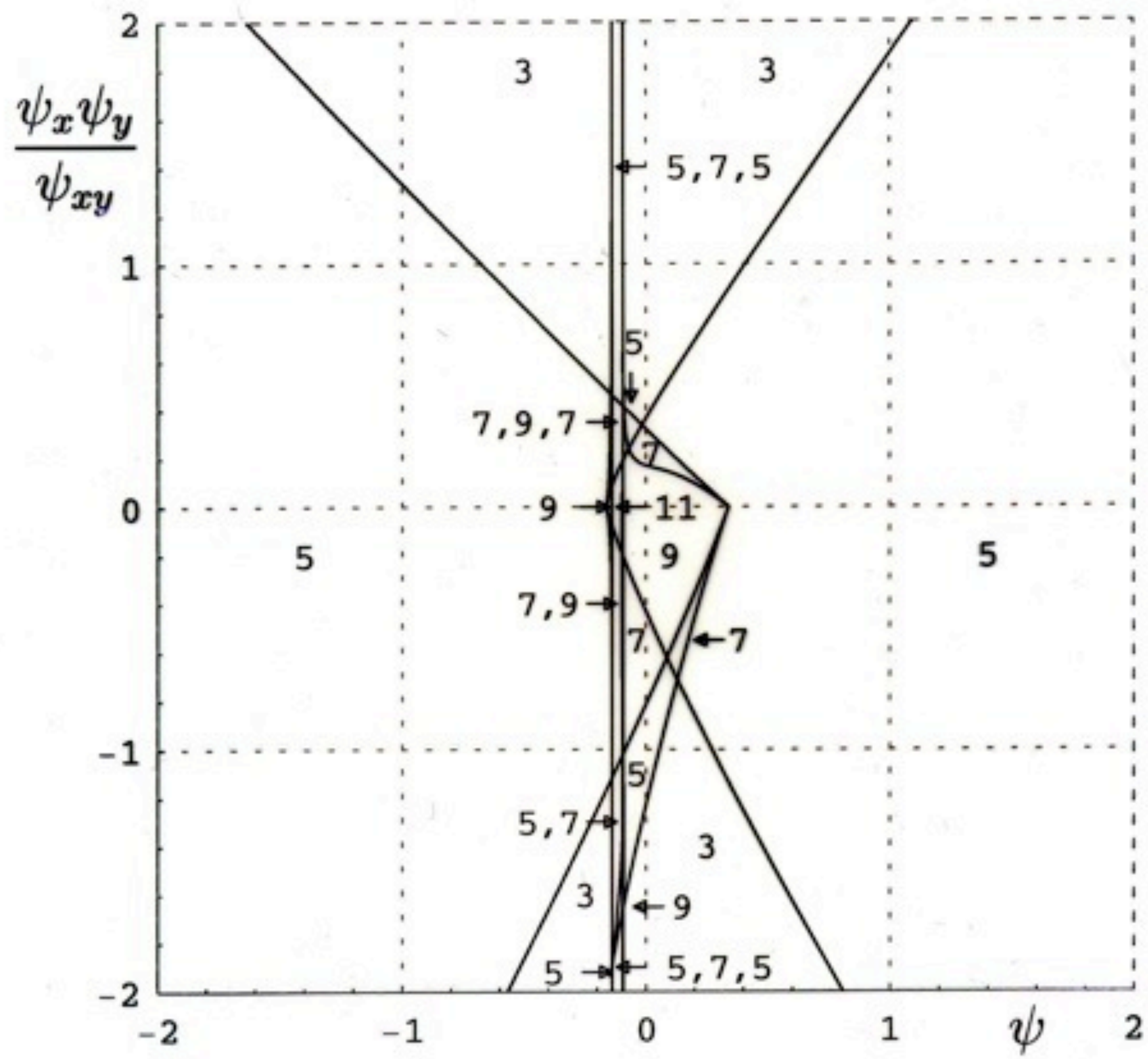


Dr. Andreas Schuppert

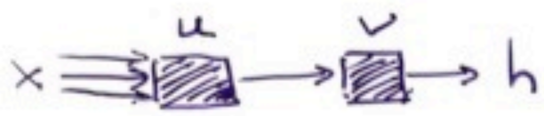
Stefan Liebscher

$$\psi = u^3 + \frac{1}{10}u^2v - uv^2 + v^3 + u^2 + uv - v^2$$

Multiplicities



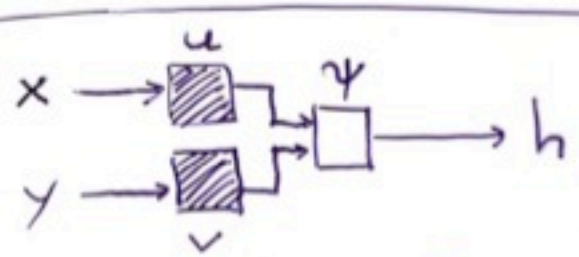
Calibrations



$v(u(x)) = h(x)$

Trivial calibration: Let u, v be solutions, $\sigma : \mathbb{R} \to \mathbb{R}$ invertible. Then

$$\left.\begin{aligned} \tilde{u} &:= \sigma^{-1} \circ u \\ \tilde{v} &:= v \circ \sigma \end{aligned}\right\} \quad \text{also solutions.}$$



$\psi(u(x), v(y)) = h(x,y)$

Nontrivial calibration: Let $\psi(u,v) = u+v$, $\sigma \in \mathbb{R}$. If u, v solve, then

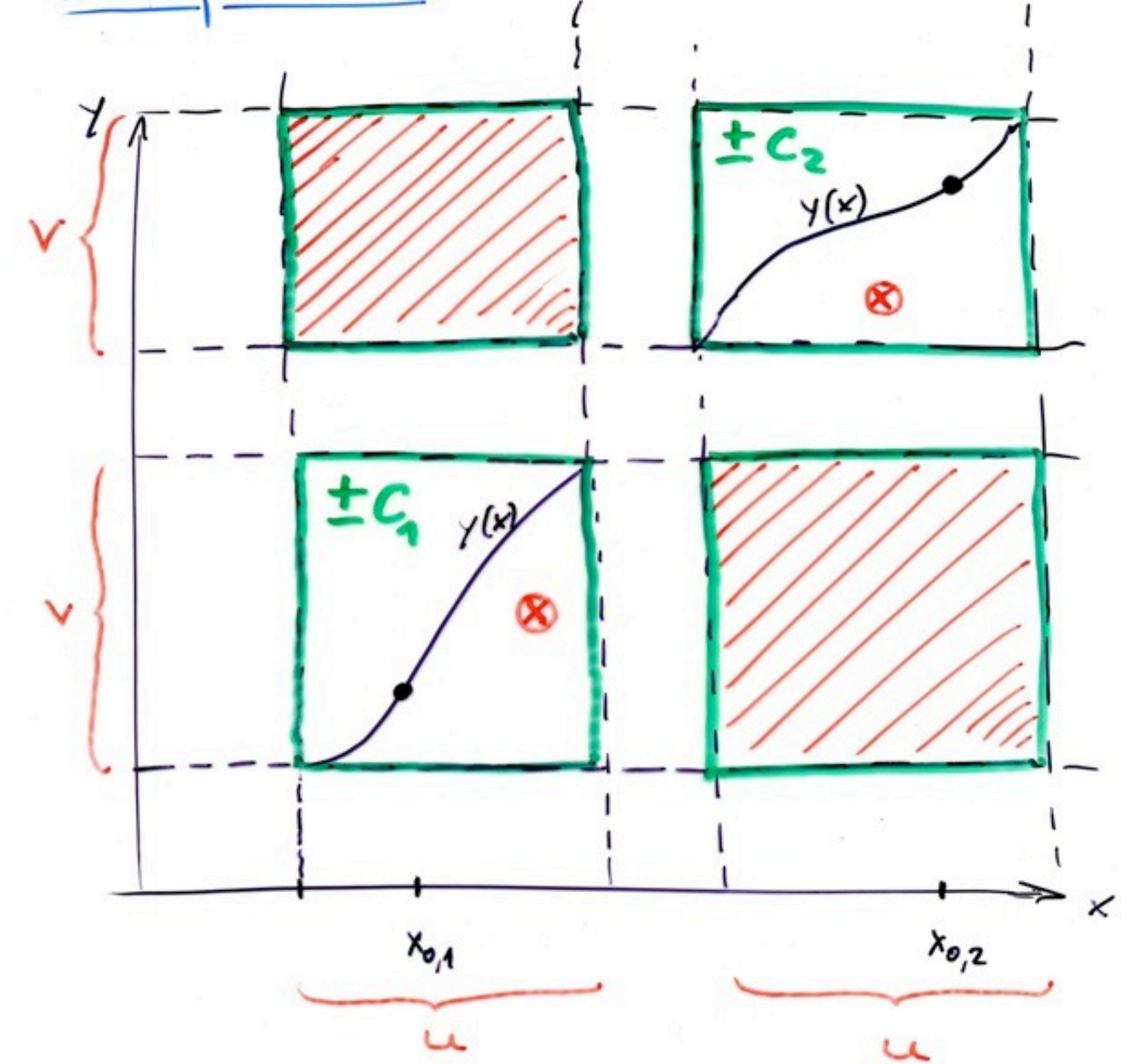
$u + \sigma$, $v - \sigma$ also solve.

Proposition: A 1-parameter family of calibrations σ exists, iff ψ satisfies some PDE

$$\alpha(u)\psi_u + \beta(v)\psi_v = 0, \quad \text{i.e.}$$

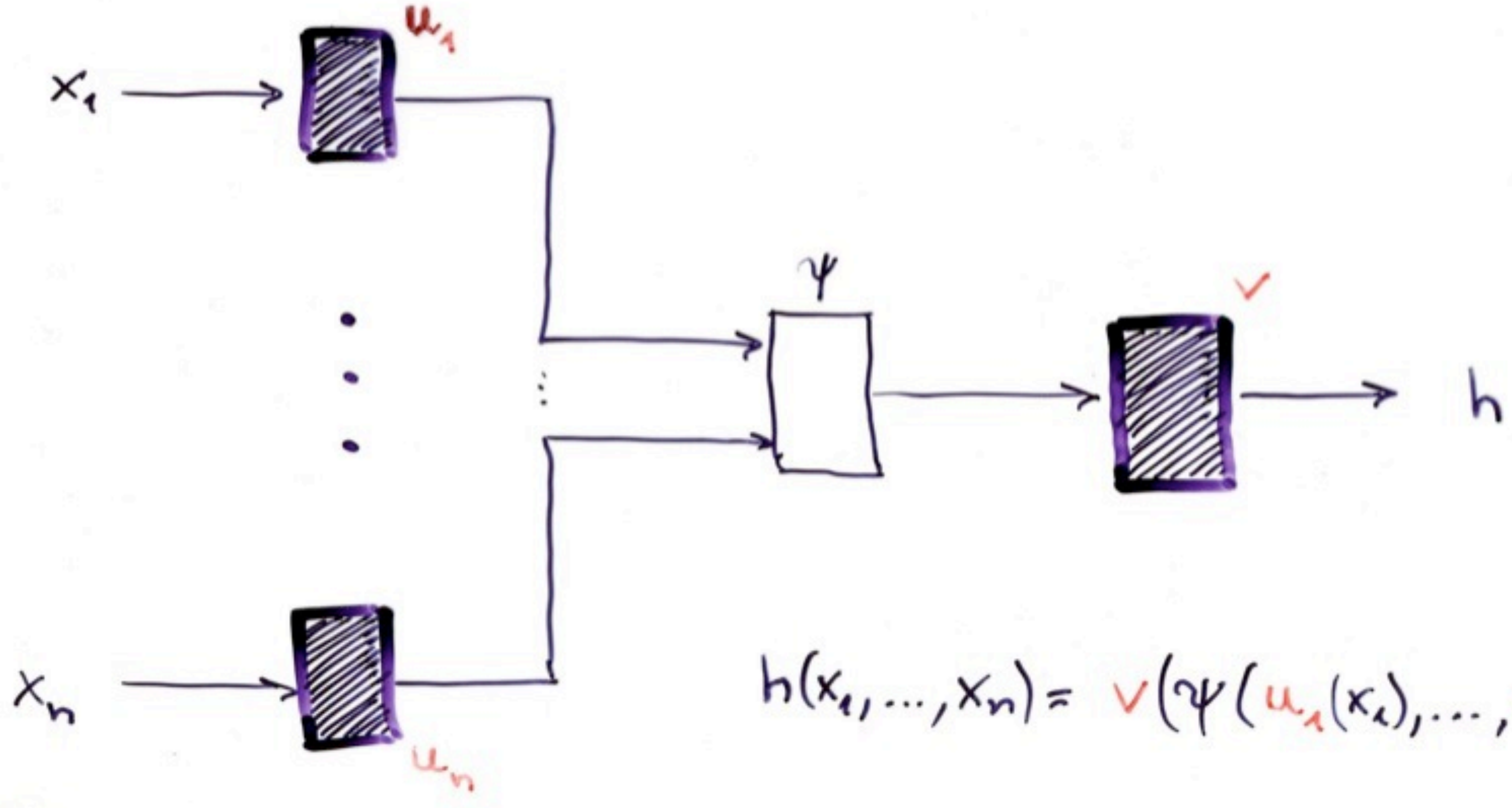
$$\psi(u,v) = f(a(u) + b(v)).$$

Extrapolation



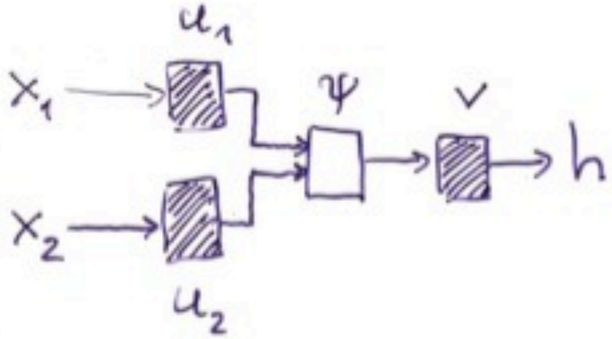
Example:

black-white-black tree (bwb)



$$h(x_1, \dots, x_n) = v(\psi(u_1(x_1), \dots, u_n(x_n)))$$

Normal form of black-white-black tree



Normal form Lemma:

Assume monotonicity $\psi_{u_1}, \psi_{u_2} \neq 0$.
Then there exist transformations u_1, u_2, v such that

$$H(x_1, x_2) = v(\psi(u_1(x_1), u_2(x_2)))$$

is in normal form:

(N1) $\quad H(t,0) = H(0,t) = t$

(N2) $\quad H(t,t) = 2t$

For uniqueness require $u_j(0) = v(0) = 0$ and normalize

(N3) $\quad D^{\underline{k}}_{\underline{x}} H(0) = 1$, (nondegeneracy)

for some multi-index $\underline{k}$ of even order.

Normal form invariants

For any multi-index $\underline{k}$, let

$$I_k(\psi, \underline{u}^0) := D_{\underline{x}}^{\underline{k}} H(0)$$

denote the Taylor coefficient of the unique normal form H of ψ.

Then the black boxes $\underline{u} = (u_1, \dots, u_n)$ of the black-white-black tree satisfy

$$\boxed{I_{\underline{k}}(\psi, \underline{u}) = I_{\underline{k}}(h, \underline{x})}$$

Proof: The I_k are invariants under the group of transformations $(u_1, \dots, u_n, v)$. $\boxtimes$

$n=3$

$$H_{xyz} = \frac{v'(0)^2}{24\psi_1^3\psi_2^3\psi_3^3} \cdot$$

$$\cdot \sum_{\text{perm}} (3\psi_1^2\psi_2^2\psi_3^2\psi_{123} -$$

$$-\psi_1\psi_2\psi_3(\psi_1^2\psi_{23}^2 + 2\psi_1\psi_2\psi_{13}\psi_{23}) -$$

$$-3\psi_1^3\psi_2^2(\psi_3\psi_{233} - \psi_{23}\psi_{33}))$$

Notation:

$$\psi_{j_1 j_2 j_3} = \partial_{u_{j_1}} \partial_{u_{j_2}} \partial_{u_{j_3}} \psi$$

$n=2$

$$H_{xxy} = \frac{v'(0)^2}{4\psi_1^2\psi_2^2}\cdot$$

$$\cdot(\psi_1^2(\psi_{12}\psi_{22} - \psi_2\psi_{122}) - \{12\})$$

$$H_{xxyy} = \frac{v'(0)^3}{14\psi_1^3\psi_2^3}\cdot$$

$$\cdot(-3\psi_1^4\psi_{12}\psi_{22}^2 +$$

$$+\psi_1^3\psi_2(3\psi_1\psi_{22}\psi_{122} +$$

$$+\psi_1\psi_{12}\psi_{222} + 2\psi_{12}^2\psi_{22}) +$$

$$+\psi_1^2\psi_2^2(\psi_{11}\psi_{22} + \psi_{1222} -$$

$$-2\psi_1\psi_{22}\psi_{112} - 2\psi_1\psi_{12}\psi_{122}) +$$

$$+\{12\})$$

Dimensions of normal form spaces

$p \backslash n$	2	3	4	5
0	0	0	0	0
1	0	0	0	0
2	0	1	4	8
3	1	7	19	37
4	2	18	49	101
5	5	35	100	221

Dimensions of spaces of normal form invariants $I_{\underline{k}}$ of order $|\underline{k}| \leq p$ for black-white-black trees

$$v(\psi(u_1(x_1),\dots,u_n(x_n))) = h(x_1,\dots,x_n)$$

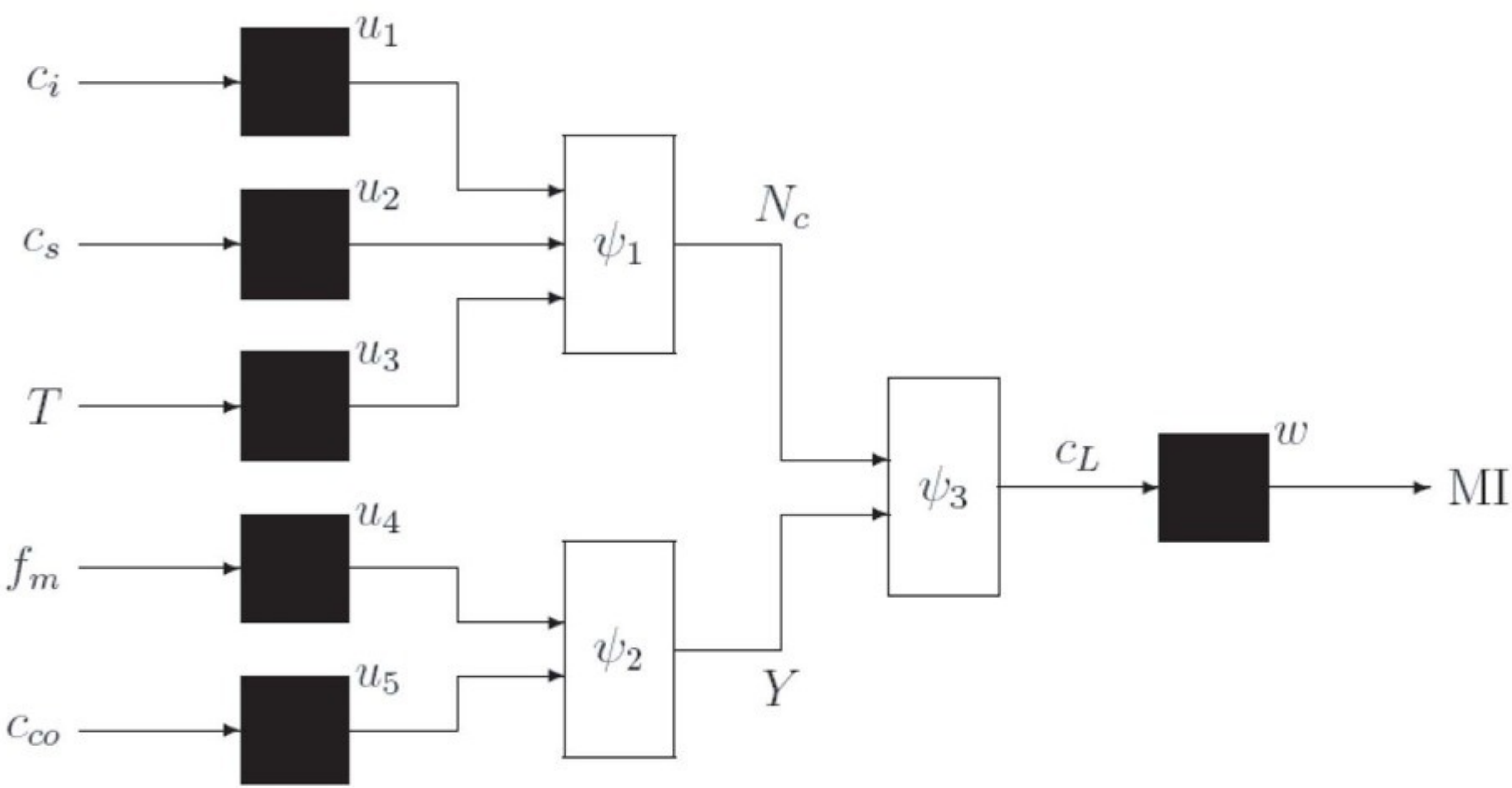
Identification theorem

Assume

- tree hybrid model $x_j = u_j((x_i)_{i \in p_j}) \in \mathbb{R}$, $j \in B \cup W$.
- monotonicity: all $u_{j,x_i} \neq 0$.
- 5-jet of data $x_N = h(x^0)$ given on data base D of general position.
- $\boxed{\dim D = \max_{j \in B} |p_j|}$ = max # black box inputs
- genericity of white boxes u_j on their inputs.

Theorem: [Fiedler & Schuppert]

All black box functions are determined uniquely on their input data, up to trivial calibration.



SHM for polymerization

Example: Kolmogorov graph

$$h(x_1,\dots,x_n) = \psi_0\Big(\big(v_j\big(\psi_j\big((u_{ji}(x_i))_{i=1,\dots,n}\big)\big)\big)_{j=1,\dots,n'}\Big)$$ $n' \geq n$, scalar.

rank condition $\boxed{\frac{1}{n!}(N+1)\cdot\ldots\cdot(N+n) > (N+1)\cdot n'\cdot\Big(1+\frac{1}{M}\sum_{i=1}^{n} M_i\Big)}$ $M = M_1\cdot\ldots\cdot M_n$

for N-jet spaces.

sufficient $(N+1)\,n'\cdot\frac{1}{M}\sum_{i=1}^{n} M_i$ small, and

$\frac{1}{n!}(N+2)\cdot\ldots\cdot(N+n) > n'$.

worst case $n=2$ (maximal branching), $N+2 \geq 2n'+1$.

[Kolmogorov] $n' = 2n+1 = 5$, $\boxed{N \geq 9.}$

```
In[60]:= dh[x1_, x2_] = D[h[x1, x2], x1];
         dh00 = ToString[Expand[dh[0, 0]], FormatType → InputForm];
         dh00 =
          StringReplace[dh00, {"[0, 0, 0, 0, 0]" → "", "[0, 0]" -> "", "[0]" → ""}];
         Print[1 + Count[Characters[dh00], "+"], "   terms"];
         dh00out = ToExpression[dh00]
```

5 terms

Out[60]=
$$u51'\,v5'\,p5^{(1,0)}\,p0^{(0,0,0,0,1)}+u41'\,v4'\,p4^{(1,0)}\,p0^{(0,0,0,1,0)}+u31'\,v3'\,p3^{(1,0)}\,p0^{(0,0,1,0,0)}+u21'\,v2'\,p2^{(1,0)}\,p0^{(0,1,0,0,0)}+u11'\,v1'\,p1^{(1,0)}\,p0^{(1,0,0,0,0)}$$

```
In[61]:= dh[x1_, x2_] = D[h[x1, x2], x1, x2];
         dh00 = ToString[Expand[dh[0, 0]], FormatType → InputForm];
         dh00 =
          StringReplace[dh00, {"[0, 0, 0, 0, 0]" → "", "[0, 0]" -> "", "[0]" → ""}];
         Print[1 + Count[Characters[dh00], "+"], "   terms"];
         dh00out = ToExpression[dh00]
```

35 terms

Out[61]=
$$\begin{aligned}&u51'\,u52'\,v5''\,p5^{(0,1)}\,p5^{(1,0)}\,p0^{(0,0,0,0,1)}+\\
&u51'\,u52'\,v5'\,p5^{(1,1)}\,p0^{(0,0,0,0,1)}+u51'\,u52'\,(v5')^2\,p5^{(0,1)}\,p5^{(1,0)}\,p0^{(0,0,0,0,2)}+\\
&u41'\,u42'\,v4''\,p4^{(0,1)}\,p4^{(1,0)}\,p0^{(0,0,0,1,0)}+u41'\,u42'\,v4'\,p4^{(1,1)}\,p0^{(0,0,0,1,0)}+\\
&u41'\,u52'\,v4'\,v5'\,p5^{(0,1)}\,p4^{(1,0)}\,p0^{(0,0,0,1,1)}+u42'\,u51'\,v4'\,v5'\,p4^{(0,1)}\,p5^{(1,0)}\,p0^{(0,0,0,1,1)}+\\
&u41'\,u42'\,(v4')^2\,p4^{(0,1)}\,p4^{(1,0)}\,p0^{(0,0,0,2,0)}+u31'\,u32'\,v3''\,p3^{(0,1)}\,p3^{(1,0)}\,p0^{(0,0,1,0,0)}+\\
&u31'\,u32'\,v3'\,p3^{(1,1)}\,p0^{(0,0,1,0,0)}+u31'\,u52'\,v3'\,v5'\,p5^{(0,1)}\,p3^{(1,0)}\,p0^{(0,0,1,0,1)}+\\
&u32'\,u51'\,v3'\,v5'\,p3^{(0,1)}\,p5^{(1,0)}\,p0^{(0,0,1,0,1)}+u31'\,u42'\,v3'\,v4'\,p4^{(0,1)}\,p3^{(1,0)}\,p0^{(0,0,1,1,0)}+\\
&u32'\,u41'\,v3'\,v4'\,p3^{(0,1)}\,p4^{(1,0)}\,p0^{(0,0,1,1,0)}+u31'\,u32'\,(v3')^2\,p3^{(0,1)}\,p3^{(1,0)}\,p0^{(0,0,2,0,0)}+\\
&u21'\,u22'\,v2''\,p2^{(0,1)}\,p2^{(1,0)}\,p0^{(0,1,0,0,0)}+u21'\,u22'\,v2'\,p2^{(1,1)}\,p0^{(0,1,0,0,0)}+\\
&u21'\,u52'\,v2'\,v5'\,p5^{(0,1)}\,p2^{(1,0)}\,p0^{(0,1,0,0,1)}+u22'\,u51'\,v2'\,v5'\,p2^{(0,1)}\,p5^{(1,0)}\,p0^{(0,1,0,0,1)}+\\
&u21'\,u42'\,v2'\,v4'\,p4^{(0,1)}\,p2^{(1,0)}\,p0^{(0,1,0,1,0)}+u22'\,u41'\,v2'\,v4'\,p2^{(0,1)}\,p4^{(1,0)}\,p0^{(0,1,0,1,0)}+\\
&u21'\,u32'\,v2'\,v3'\,p3^{(0,1)}\,p2^{(1,0)}\,p0^{(0,1,1,0,0)}+u22'\,u31'\,v2'\,v3'\,p2^{(0,1)}\,p3^{(1,0)}\,p0^{(0,1,1,0,0)}+\\
&u21'\,u22'\,(v2')^2\,p2^{(0,1)}\,p2^{(1,0)}\,p0^{(0,2,0,0,0)}+u11'\,u12'\,v1''\,p1^{(0,1)}\,p1^{(1,0)}\,p0^{(1,0,0,0,0)}+\\
&u11'\,u12'\,v1'\,p1^{(1,1)}\,p0^{(1,0,0,0,0)}+u11'\,u52'\,v1'\,v5'\,p5^{(0,1)}\,p1^{(1,0)}\,p0^{(1,0,0,0,1)}+\\
&u12'\,u51'\,v1'\,v5'\,p1^{(0,1)}\,p5^{(1,0)}\,p0^{(1,0,0,0,1)}+u11'\,u42'\,v1'\,v4'\,p4^{(0,1)}\,p1^{(1,0)}\,p0^{(1,0,0,1,0)}+\\
&u12'\,u41'\,v1'\,v4'\,p1^{(0,1)}\,p4^{(1,0)}\,p0^{(1,0,0,1,0)}+u11'\,u32'\,v1'\,v3'\,p3^{(0,1)}\,p1^{(1,0)}\,p0^{(1,0,1,0,0)}+\\
&u12'\,u31'\,v1'\,v3'\,p1^{(0,1)}\,p3^{(1,0)}\,p0^{(1,0,1,0,0)}+u11'\,u22'\,v1'\,v2'\,p2^{(0,1)}\,p1^{(1,0)}\,p0^{(1,1,0,0,0)}+\\
&u12'\,u21'\,v1'\,v2'\,p1^{(0,1)}\,p2^{(1,0)}\,p0^{(1,1,0,0,0)}+u11'\,u12'\,(v1')^2\,p1^{(0,1)}\,p1^{(1,0)}\,p0^{(2,0,0,0,0)}\end{aligned}$$

```
In[62]:= dh[x1_, x2_] = D[h[x1, x2], x1, x2, x1];
         dh00 = ToString[Expand[dh[0, 0]], FormatType → InputForm];
         dh00 =
          StringReplace[dh00, {"[0, 0, 0, 0, 0]" → "", "[0, 0]" -> "", "[0]" → ""}];
         Print[1 + Count[Characters[dh00], "+"], "   terms"];
         dh00out = ToExpression[dh00]
```

225 terms

Out[62]=
$$\begin{aligned}&u52'\,u51''\,v5''\,p5^{(0,1)}\,p5^{(1,0)}\,p0^{(0,0,0,0,1)}+\\
&(u51')^2\,u52'\,v5^{(3)}\,p5^{(0,1)}\,(p5^{(1,0)})^2\,p0^{(0,0,0,0,1)}+u52'\,v5'\,u51''\,p5^{(1,1)}\,p0^{(0,0,0,0,1)}+\\
&2\,(u51')^2\,u52'\,v5''\,p5^{(1,0)}\,p5^{(1,1)}\,p0^{(0,0,0,0,1)}+(u51')^2\,u52'\,v5''\,p5^{(0,1)}\,p5^{(2,0)}\,p0^{(0,0,0,0,1)}+\\
&(u51')^2\,u52'\,v5'\,p5^{(2,1)}\,p0^{(0,0,0,0,1)}+u52'\,(v5')^2\,u51''\,p5^{(0,1)}\,p5^{(1,0)}\,p0^{(0,0,0,0,2)}+\\
&3\,(u51')^2\,u52'\,v5'\,v5''\,p5^{(0,1)}\,(p5^{(1,0)})^2\,p0^{(0,0,0,0,2)}+\\
&2\,(u51')^2\,u52'\,(v5')^2\,p5^{(1,0)}\,p5^{(1,1)}\,p0^{(0,0,0,0,2)}+(u51')^2\,u52'\,(v5')^2\,p5^{(0,1)}\,p5^{(2,0)}\,p0^{(0,0,0,0,2)}+\\
&(u51')^2\,u52'\,(v5')^3\,p5^{(0,1)}\,(p5^{(1,0)})^2\,p0^{(0,0,0,0,3)}+u42'\,u41''\,v4''\,p4^{(0,1)}\,p4^{(1,0)}\,p0^{(0,0,0,1,0)}+\\
&(u41')^2\,u42'\,v4^{(3)}\,p4^{(0,1)}\,(p4^{(1,0)})^2\,p0^{(0,0,0,1,0)}+u42'\,v4'\,u41''\,p4^{(1,1)}\,p0^{(0,0,0,1,0)}+\\
&2\,(u41')^2\,u42'\,v4''\,p4^{(1,0)}\,p4^{(1,1)}\,p0^{(0,0,0,1,0)}+(u41')^2\,u42'\,v4''\,p4^{(0,1)}\,p4^{(2,0)}\,p0^{(0,0,0,1,0)}+\\
&(u41')^2\,u42'\,v4'\,p4^{(2,1)}\,p0^{(0,0,0,1,0)}+u52'\,v4'\,v5'\,u41''\,p5^{(0,1)}\,p4^{(1,0)}\,p0^{(0,0,0,1,1)}+\end{aligned}$$

$(u41')^2\ u52'\ v5'\ v4''\ p5^{(0,1)}\ (p4^{(1,0)})^2\ p0^{(0,0,0,1,1)} + u42'\ v4'\ v5'\ u51''\ p4^{(0,1)}\ p5^{(1,0)}\ p0^{(0,0,0,1,1)} +$

$2\ u41'\ u42'\ u51'\ v5'\ v4''\ p4^{(0,1)}\ p4^{(1,0)}\ p5^{(1,0)}\ p0^{(0,0,0,1,1)} +$

$2\ u41'\ u51'\ u52'\ v4'\ v5''\ p5^{(0,1)}\ p4^{(1,0)}\ p5^{(1,0)}\ p0^{(0,0,0,1,1)} +$

$u42'\ (u51')^2\ v4'\ v5''\ p4^{(0,1)}\ (p5^{(1,0)})^2\ p0^{(0,0,0,1,1)} +$

$2\ u41'\ u42'\ u51'\ v4'\ v5'\ p5^{(1,0)}\ p4^{(1,1)}\ p0^{(0,0,0,1,1)} +$

$2\ u41'\ u51'\ u52'\ v4'\ v5'\ p4^{(1,0)}\ p5^{(1,1)}\ p0^{(0,0,0,1,1)} +$

$(u41')^2\ u52'\ v4'\ v5'\ p5^{(0,1)}\ p4^{(2,0)}\ p0^{(0,0,0,1,1)} + u42'\ (u51')^2\ v4'\ v5'\ p4^{(0,1)}\ p5^{(2,0)}\ p0^{(0,0,0,1,1)} +$

$2\ u41'\ u51'\ u52'\ v4'\ (v5')^2\ p5^{(0,1)}\ p4^{(1,0)}\ p5^{(1,0)}\ p0^{(0,0,0,1,2)} + u42'\ (u51')^2\ v4'$

$(v5')^2\ p4^{(0,1)}\ (p5^{(1,0)})^2\ p0^{(0,0,0,1,2)} + u42'\ (v4')^2\ u41''\ p4^{(0,1)}\ p4^{(1,0)}\ p0^{(0,0,0,2,0)} +$

$3\ (u41')^2\ u42'\ v4'\ v4''\ p4^{(0,1)}\ (p4^{(1,0)})^2\ p0^{(0,0,0,2,0)} +$

$2\ (u41')^2\ u42'\ (v4')^2\ p4^{(1,0)}\ p4^{(1,1)}\ p0^{(0,0,0,2,0)} + (u41')^2\ u42'\ (v4')^2\ p4^{(0,1)}\ p4^{(2,0)}\ p0^{(0,0,0,2,0)} +$

$(u41')^2\ u52'\ (v4')^2\ v5'\ p5^{(0,1)}\ (p4^{(1,0)})^2\ p0^{(0,0,0,2,1)} +$

$2\ u41'\ u42'\ u51'\ (v4')^2\ v5'\ p4^{(0,1)}\ p4^{(1,0)}\ p5^{(1,0)}\ p0^{(0,0,0,2,1)} +$

$(u41')^2\ u42'\ (v4')^3\ p4^{(0,1)}\ (p4^{(1,0)})^2\ p0^{(0,0,0,3,0)} + u32'\ u31''\ v3''\ p3^{(0,1)}\ p3^{(1,0)}\ p0^{(0,0,1,0,0)} +$

$(u31')^2\ u32'\ v3^{(3)}\ p3^{(0,1)}\ (p3^{(1,0)})^2\ p0^{(0,0,1,0,0)} + u32'\ v3'\ u31''\ p3^{(1,1)}\ p0^{(0,0,1,0,0)} +$

$2\ (u31')^2\ u32'\ v3''\ p3^{(1,0)}\ p3^{(1,1)}\ p0^{(0,0,1,0,0)} + (u31')^2\ u32'\ v3''\ p3^{(0,1)}\ p3^{(2,0)}\ p0^{(0,0,1,0,0)} +$

$(u31')^2\ u32'\ v3'\ p3^{(2,1)}\ p0^{(0,0,1,0,0)} + u52'\ v3'\ v5'\ u31''\ p5^{(0,1)}\ p3^{(1,0)}\ p0^{(0,0,1,0,1)} +$

$(u31')^2\ u52'\ v5'\ v3''\ p5^{(0,1)}\ (p3^{(1,0)})^2\ p0^{(0,0,1,0,1)} + u32'\ v3'\ v5'\ u51''\ p3^{(0,1)}\ p5^{(1,0)}\ p0^{(0,0,1,0,1)} +$

$2\ u31'\ u32'\ u51'\ v5'\ v3''\ p3^{(0,1)}\ p3^{(1,0)}\ p5^{(1,0)}\ p0^{(0,0,1,0,1)} +$

$2\ u31'\ u51'\ u52'\ v3'\ v5''\ p5^{(0,1)}\ p3^{(1,0)}\ p5^{(1,0)}\ p0^{(0,0,1,0,1)} +$

$u32'\ (u51')^2\ v3'\ v5''\ p3^{(0,1)}\ (p5^{(1,0)})^2\ p0^{(0,0,1,0,1)} +$

$2\ u31'\ u32'\ u51'\ v3'\ v5'\ p5^{(1,0)}\ p3^{(1,1)}\ p0^{(0,0,1,0,1)} +$

$2\ u31'\ u51'\ u52'\ v3'\ v5'\ p3^{(1,0)}\ p5^{(1,1)}\ p0^{(0,0,1,0,1)} +$

$(u31')^2\ u52'\ v3'\ v5'\ p5^{(0,1)}\ p3^{(2,0)}\ p0^{(0,0,1,0,1)} + u32'\ (u51')^2\ v3'\ v5'\ p3^{(0,1)}\ p5^{(2,0)}\ p0^{(0,0,1,0,1)} +$

$2\ u31'\ u51'\ u52'\ v3'\ (v5')^2\ p5^{(0,1)}\ p3^{(1,0)}\ p5^{(1,0)}\ p0^{(0,0,1,0,2)} + u32'\ (u51')^2\ v3'$

$(v5')^2\ p3^{(0,1)}\ (p5^{(1,0)})^2\ p0^{(0,0,1,0,2)} + u42'\ v3'\ v4'\ u31''\ p4^{(0,1)}\ p3^{(1,0)}\ p0^{(0,0,1,1,0)} +$

$(u31')^2\ u42'\ v4'\ v3''\ p4^{(0,1)}\ (p3^{(1,0)})^2\ p0^{(0,0,1,1,0)} + u32'\ v3'\ v4'\ u41''\ p3^{(0,1)}\ p4^{(1,0)}\ p0^{(0,0,1,1,0)} +$

$2\ u31'\ u32'\ u41'\ v4'\ v3''\ p3^{(0,1)}\ p3^{(1,0)}\ p4^{(1,0)}\ p0^{(0,0,1,1,0)} +$

$2\ u31'\ u41'\ u42'\ v3'\ v4''\ p4^{(0,1)}\ p3^{(1,0)}\ p4^{(1,0)}\ p0^{(0,0,1,1,0)} +$

$u32'\ (u41')^2\ v3'\ v4''\ p3^{(0,1)}\ (p4^{(1,0)})^2\ p0^{(0,0,1,1,0)} +$

$2\ u31'\ u32'\ u41'\ v3'\ v4'\ p4^{(1,0)}\ p3^{(1,1)}\ p0^{(0,0,1,1,0)} +$

$2\ u31'\ u41'\ u42'\ v3'\ v4'\ p3^{(1,0)}\ p4^{(1,1)}\ p0^{(0,0,1,1,0)} +$

$(u31')^2\ u42'\ v3'\ v4'\ p4^{(0,1)}\ p3^{(2,0)}\ p0^{(0,0,1,1,0)} + u32'\ (u41')^2\ v3'\ v4'\ p3^{(0,1)}\ p4^{(2,0)}\ p0^{(0,0,1,1,0)} +$

$2\ u31'\ u41'\ u52'\ v3'\ v4'\ v5'\ p5^{(0,1)}\ p3^{(1,0)}\ p4^{(1,0)}\ p0^{(0,0,1,1,1)} +$

$2\ u31'\ u42'\ u51'\ v3'\ v4'\ v5'\ p4^{(0,1)}\ p3^{(1,0)}\ p5^{(1,0)}\ p0^{(0,0,1,1,1)} +$

$2\ u32'\ u41'\ u51'\ v3'\ v4'\ v5'\ p3^{(0,1)}\ p4^{(1,0)}\ p5^{(1,0)}\ p0^{(0,0,1,1,1)} +$

$2\ u31'\ u41'\ u42'\ v3'\ (v4')^2\ p4^{(0,1)}\ p3^{(1,0)}\ p4^{(1,0)}\ p0^{(0,0,1,2,0)} + u32'\ (u41')^2\ v3'$

$(v4')^2\ p3^{(0,1)}\ (p4^{(1,0)})^2\ p0^{(0,0,1,2,0)} + u32'\ (v3')^2\ u31''\ p3^{(0,1)}\ p3^{(1,0)}\ p0^{(0,0,2,0,0)} +$

$3\ (u31')^2\ u32'\ v3'\ v3''\ p3^{(0,1)}\ (p3^{(1,0)})^2\ p0^{(0,0,2,0,0)} +$

$2\ (u31')^2\ u32'\ (v3')^2\ p3^{(1,0)}\ p3^{(1,1)}\ p0^{(0,0,2,0,0)} + (u31')^2\ u32'\ (v3')^2\ p3^{(0,1)}\ p3^{(2,0)}\ p0^{(0,0,2,0,0)} +$

$(u31')^2\ u52'\ (v3')^2\ v5'\ p5^{(0,1)}\ (p3^{(1,0)})^2\ p0^{(0,0,2,0,1)} +$

$2\ u31'\ u32'\ u51'\ (v3')^2\ v5'\ p3^{(0,1)}\ p3^{(1,0)}\ p5^{(1,0)}\ p0^{(0,0,2,0,1)} +$

$(u31')^2\ u42'\ (v3')^2\ v4'\ p4^{(0,1)}\ (p3^{(1,0)})^2\ p0^{(0,0,2,1,0)} +$

$2\ u31'\ u32'\ u41'\ (v3')^2\ v4'\ p3^{(0,1)}\ p3^{(1,0)}\ p4^{(1,0)}\ p0^{(0,0,2,1,0)} +$

$(u31')^2\ u32'\ (v3')^3\ p3^{(0,1)}\ (p3^{(1,0)})^2\ p0^{(0,0,3,0,0)} + u22'\ u21''\ v2''\ p2^{(0,1)}\ p2^{(1,0)}\ p0^{(0,1,0,0,0)} +$

$(u21')^2\ u22'\ v2^{(3)}\ p2^{(0,1)}\ (p2^{(1,0)})^2\ p0^{(0,1,0,0,0)} + u22'\ v2'\ u21''\ p2^{(1,1)}\ p0^{(0,1,0,0,0)} +$

$2\ (u21')^2\ u22'\ v2''\ p2^{(1,0)}\ p2^{(1,1)}\ p0^{(0,1,0,0,0)} + (u21')^2\ u22'\ v2''\ p2^{(0,1)}\ p2^{(2,0)}\ p0^{(0,1,0,0,0)} +$

$(u21')^2\ u22'\ v2'\ p2^{(2,1)}\ p0^{(0,1,0,0,0)} + u52'\ v2'\ v5'\ u21''\ p5^{(0,1)}\ p2^{(1,0)}\ p0^{(0,1,0,0,1)} +$

$(u21')^2\ u52'\ v5'\ v2''\ p5^{(0,1)}\ (p2^{(1,0)})^2\ p0^{(0,1,0,0,1)} + u22'\ v2'\ v5'\ u51''\ p2^{(0,1)}\ p5^{(1,0)}\ p0^{(0,1,0,0,1)} +$

$2\ u21'\ u22'\ u51'\ v5'\ v2''\ p2^{(0,1)}\ p2^{(1,0)}\ p5^{(1,0)}\ p0^{(0,1,0,0,1)} +$

$2\ u21'\ u51'\ u52'\ v2'\ v5''\ p5^{(0,1)}\ p2^{(1,0)}\ p5^{(1,0)}\ p0^{(0,1,0,0,1)} +$

$u22'\ (u51')^2\ v2'\ v5''\ p2^{(0,1)}\ (p5^{(1,0)})^2\ p0^{(0,1,0,0,1)} +$

$2\ u21'\ u22'\ u51'\ v2'\ v5'\ p5^{(1,0)}\ p2^{(1,1)}\ p0^{(0,1,0,0,1)} +$

$2\ u21'\ u51'\ u52'\ v2'\ v5'\ p2^{(1,0)}\ p5^{(1,1)}\ p0^{(0,1,0,0,1)} +$

$(u21')^2\, u52'\, v2'\, v5'\, p5^{(0,1)}\, p2^{(2,0)}\, p0^{(0,1,0,0,1)} + u22'\, (u51')^2\, v2'\, v5'\, p2^{(0,1)}\, p5^{(2,0)}\, p0^{(0,1,0,0,1)} +$
$2\, u21'\, u51'\, u52'\, v2'\, (v5')^2\, p5^{(0,1)}\, p2^{(1,0)}\, p5^{(1,0)}\, p0^{(0,1,0,0,2)} + u22'\, (u51')^2\, v2'$
$(v5')^2\, p2^{(0,1)}\, (p5^{(1,0)})^2\, p0^{(0,1,0,0,2)} + u42'\, v2'\, v4'\, u21''\, p4^{(0,1)}\, p2^{(1,0)}\, p0^{(0,1,0,1,0)} +$
$(u21')^2\, u42'\, v4'\, v2''\, p4^{(0,1)}\, (p2^{(1,0)})^2\, p0^{(0,1,0,1,0)} + u22'\, v2'\, v4'\, u41''\, p2^{(0,1)}\, p4^{(1,0)}\, p0^{(0,1,0,1,0)} +$
$2\, u21'\, u22'\, u41'\, v4'\, v2''\, p2^{(0,1)}\, p2^{(1,0)}\, p4^{(1,0)}\, p0^{(0,1,0,1,0)} +$
$2\, u21'\, u41'\, u42'\, v2'\, v4''\, p4^{(0,1)}\, p2^{(1,0)}\, p4^{(1,0)}\, p0^{(0,1,0,1,0)} +$
$u22'\, (u41')^2\, v2'\, v4''\, p2^{(0,1)}\, (p4^{(1,0)})^2\, p0^{(0,1,0,1,0)} +$
$2\, u21'\, u22'\, u41'\, v2'\, v4'\, p4^{(1,0)}\, p2^{(1,1)}\, p0^{(0,1,0,1,0)} +$
$2\, u21'\, u41'\, u42'\, v2'\, v4'\, p2^{(1,0)}\, p4^{(1,1)}\, p0^{(0,1,0,1,0)} +$
$(u21')^2\, u42'\, v2'\, v4'\, p4^{(0,1)}\, p2^{(2,0)}\, p0^{(0,1,0,1,0)} + u22'\, (u41')^2\, v2'\, v4'\, p2^{(0,1)}\, p4^{(2,0)}\, p0^{(0,1,0,1,0)} +$
$2\, u21'\, u41'\, u52'\, v2'\, v4'\, v5'\, p5^{(0,1)}\, p2^{(1,0)}\, p4^{(1,0)}\, p0^{(0,1,0,1,1)} +$
$2\, u21'\, u42'\, u51'\, v2'\, v4'\, v5'\, p4^{(0,1)}\, p2^{(1,0)}\, p5^{(1,0)}\, p0^{(0,1,0,1,1)} +$
$2\, u22'\, u41'\, u51'\, v2'\, v4'\, v5'\, p2^{(0,1)}\, p4^{(1,0)}\, p5^{(1,0)}\, p0^{(0,1,0,1,1)} +$
$2\, u21'\, u41'\, u42'\, v2'\, (v4')^2\, p4^{(0,1)}\, p2^{(1,0)}\, p4^{(1,0)}\, p0^{(0,1,0,2,0)} + u22'\, (u41')^2\, v2'$
$(v4')^2\, p2^{(0,1)}\, (p4^{(1,0)})^2\, p0^{(0,1,0,2,0)} + u32'\, v2'\, v3'\, u21''\, p3^{(0,1)}\, p2^{(1,0)}\, p0^{(0,1,1,0,0)} +$
$(u21')^2\, u32'\, v3'\, v2''\, p3^{(0,1)}\, (p2^{(1,0)})^2\, p0^{(0,1,1,0,0)} + u22'\, v2'\, v3'\, u31''\, p2^{(0,1)}\, p3^{(1,0)}\, p0^{(0,1,1,0,0)} +$
$2\, u21'\, u22'\, u31'\, v3'\, v2''\, p2^{(0,1)}\, p2^{(1,0)}\, p3^{(1,0)}\, p0^{(0,1,1,0,0)} +$
$2\, u21'\, u31'\, u32'\, v2'\, v3''\, p3^{(0,1)}\, p2^{(1,0)}\, p3^{(1,0)}\, p0^{(0,1,1,0,0)} +$
$u22'\, (u31')^2\, v2'\, v3''\, p2^{(0,1)}\, (p3^{(1,0)})^2\, p0^{(0,1,1,0,0)} +$
$2\, u21'\, u22'\, u31'\, v2'\, v3'\, p3^{(1,0)}\, p2^{(1,1)}\, p0^{(0,1,1,0,0)} +$
$2\, u21'\, u31'\, u32'\, v2'\, v3'\, p2^{(1,0)}\, p3^{(1,1)}\, p0^{(0,1,1,0,0)} +$
$(u21')^2\, u32'\, v2'\, v3'\, p3^{(0,1)}\, p2^{(2,0)}\, p0^{(0,1,1,0,0)} + u22'\, (u31')^2\, v2'\, v3'\, p2^{(0,1)}\, p3^{(2,0)}\, p0^{(0,1,1,0,0)} +$
$2\, u21'\, u31'\, u52'\, v2'\, v3'\, v5'\, p5^{(0,1)}\, p2^{(1,0)}\, p3^{(1,0)}\, p0^{(0,1,1,0,1)} +$
$2\, u21'\, u32'\, u51'\, v2'\, v3'\, v5'\, p3^{(0,1)}\, p2^{(1,0)}\, p5^{(1,0)}\, p0^{(0,1,1,0,1)} +$
$2\, u22'\, u31'\, u51'\, v2'\, v3'\, v5'\, p2^{(0,1)}\, p3^{(1,0)}\, p5^{(1,0)}\, p0^{(0,1,1,0,1)} +$
$2\, u21'\, u31'\, u42'\, v2'\, v3'\, v4'\, p4^{(0,1)}\, p2^{(1,0)}\, p3^{(1,0)}\, p0^{(0,1,1,1,0)} +$
$2\, u21'\, u32'\, u41'\, v2'\, v3'\, v4'\, p3^{(0,1)}\, p2^{(1,0)}\, p4^{(1,0)}\, p0^{(0,1,1,1,0)} +$
$2\, u22'\, u31'\, u41'\, v2'\, v3'\, v4'\, p2^{(0,1)}\, p3^{(1,0)}\, p4^{(1,0)}\, p0^{(0,1,1,1,0)} +$
$2\, u21'\, u31'\, u32'\, v2'\, (v3')^2\, p3^{(0,1)}\, p2^{(1,0)}\, p3^{(1,0)}\, p0^{(0,1,2,0,0)} + u22'\, (u31')^2\, v2'$
$(v3')^2\, p2^{(0,1)}\, (p3^{(1,0)})^2\, p0^{(0,1,2,0,0)} + u22'\, (v2')^2\, u21''\, p2^{(0,1)}\, p2^{(1,0)}\, p0^{(0,2,0,0,0)} +$
$3\, (u21')^2\, u22'\, v2'\, v2''\, p2^{(0,1)}\, (p2^{(1,0)})^2\, p0^{(0,2,0,0,0)} +$
$2\, (u21')^2\, u22'\, (v2')^2\, p2^{(1,0)}\, p2^{(1,1)}\, p0^{(0,2,0,0,0)} + (u21')^2\, u22'\, (v2')^2\, p2^{(0,1)}\, p2^{(2,0)}\, p0^{(0,2,0,0,0)} +$
$(u21')^2\, u52'\, (v2')^2\, v5'\, p5^{(0,1)}\, (p2^{(1,0)})^2\, p0^{(0,2,0,0,1)} +$
$2\, u21'\, u22'\, u51'\, (v2')^2\, v5'\, p2^{(0,1)}\, p2^{(1,0)}\, p5^{(1,0)}\, p0^{(0,2,0,0,1)} +$
$(u21')^2\, u42'\, (v2')^2\, v4'\, p4^{(0,1)}\, (p2^{(1,0)})^2\, p0^{(0,2,0,1,0)} +$
$2\, u21'\, u22'\, u41'\, (v2')^2\, v4'\, p2^{(0,1)}\, p2^{(1,0)}\, p4^{(1,0)}\, p0^{(0,2,0,1,0)} +$
$(u21')^2\, u32'\, (v2')^2\, v3'\, p3^{(0,1)}\, (p2^{(1,0)})^2\, p0^{(0,2,1,0,0)} +$
$2\, u21'\, u22'\, u31'\, (v2')^2\, v3'\, p2^{(0,1)}\, p2^{(1,0)}\, p3^{(1,0)}\, p0^{(0,2,1,0,0)} +$
$(u21')^2\, u22'\, (v2')^3\, p2^{(0,1)}\, (p2^{(1,0)})^2\, p0^{(0,3,0,0,0)} + u12'\, u11''\, v1''\, p1^{(0,1)}\, p1^{(1,0)}\, p0^{(1,0,0,0,0)} +$
$(u11')^2\, u12'\, v1^{(3)}\, p1^{(0,1)}\, (p1^{(1,0)})^2\, p0^{(1,0,0,0,0)} + u12'\, v1'\, u11''\, p1^{(1,1)}\, p0^{(1,0,0,0,0)} +$
$2\, (u11')^2\, u12'\, v1''\, p1^{(1,0)}\, p1^{(1,1)}\, p0^{(1,0,0,0,0)} + (u11')^2\, u12'\, v1''\, p1^{(0,1)}\, p1^{(2,0)}\, p0^{(1,0,0,0,0)} +$
$(u11')^2\, u12'\, v1'\, p1^{(2,1)}\, p0^{(1,0,0,0,0)} + u52'\, v1'\, v5'\, u11''\, p5^{(0,1)}\, p1^{(1,0)}\, p0^{(1,0,0,0,1)} +$
$(u11')^2\, u52'\, v5'\, v1''\, p5^{(0,1)}\, (p1^{(1,0)})^2\, p0^{(1,0,0,0,1)} + u12'\, v1'\, v5'\, u51''\, p1^{(0,1)}\, p5^{(1,0)}\, p0^{(1,0,0,0,1)} +$
$2\, u11'\, u12'\, u51'\, v5'\, v1''\, p1^{(0,1)}\, p1^{(1,0)}\, p5^{(1,0)}\, p0^{(1,0,0,0,1)} +$
$2\, u11'\, u51'\, u52'\, v1'\, v5''\, p5^{(0,1)}\, p1^{(1,0)}\, p5^{(1,0)}\, p0^{(1,0,0,0,1)} +$
$u12'\, (u51')^2\, v1'\, v5''\, p1^{(0,1)}\, (p5^{(1,0)})^2\, p0^{(1,0,0,0,1)} +$
$2\, u11'\, u12'\, u51'\, v1'\, v5'\, p5^{(1,0)}\, p1^{(1,1)}\, p0^{(1,0,0,0,1)} +$
$2\, u11'\, u51'\, u52'\, v1'\, v5'\, p1^{(1,0)}\, p5^{(1,1)}\, p0^{(1,0,0,0,1)} +$
$(u11')^2\, u52'\, v1'\, v5'\, p5^{(0,1)}\, p1^{(2,0)}\, p0^{(1,0,0,0,1)} + u12'\, (u51')^2\, v1'\, v5'\, p1^{(0,1)}\, p5^{(2,0)}\, p0^{(1,0,0,0,1)} +$
$2\, u11'\, u51'\, u52'\, v1'\, (v5')^2\, p5^{(0,1)}\, p1^{(1,0)}\, p5^{(1,0)}\, p0^{(1,0,0,0,2)} + u12'\, (u51')^2\, v1'$
$(v5')^2\, p1^{(0,1)}\, (p5^{(1,0)})^2\, p0^{(1,0,0,0,2)} + u42'\, v1'\, v4'\, u11''\, p4^{(0,1)}\, p1^{(1,0)}\, p0^{(1,0,0,1,0)} +$
$(u11')^2\, u42'\, v4'\, v1''\, p4^{(0,1)}\, (p1^{(1,0)})^2\, p0^{(1,0,0,1,0)} + u12'\, v1'\, v4'\, u41''\, p1^{(0,1)}\, p4^{(1,0)}\, p0^{(1,0,0,1,0)} +$
$2\, u11'\, u12'\, u41'\, v4'\, v1''\, p1^{(0,1)}\, p1^{(1,0)}\, p4^{(1,0)}\, p0^{(1,0,0,1,0)} +$
$2\, u11'\, u41'\, u42'\, v1'\, v4''\, p4^{(0,1)}\, p1^{(1,0)}\, p4^{(1,0)}\, p0^{(1,0,0,1,0)} +$
$u12'\, (u41')^2\, v1'\, v4''\, p1^{(0,1)}\, (p4^{(1,0)})^2\, p0^{(1,0,0,1,0)} +$

$2\,u11'\,u12'\,u41'\,v1'\,v4'\,p4^{(1,0)}\,p1^{(1,1)}\,p0^{(1,0,0,1,0)}+$
$2\,u11'\,u41'\,u42'\,v1'\,v4'\,p1^{(1,0)}\,p4^{(1,1)}\,p0^{(1,0,0,1,0)}+$
$(u11')^2\,u42'\,v1'\,v4'\,p4^{(0,1)}\,p1^{(2,0)}\,p0^{(1,0,0,1,0)}+u12'\,(u41')^2\,v1'\,v4'\,p1^{(0,1)}\,p4^{(2,0)}\,p0^{(1,0,0,1,0)}+$
$2\,u11'\,u41'\,u52'\,v1'\,v4'\,v5'\,p5^{(0,1)}\,p1^{(1,0)}\,p4^{(1,0)}\,p0^{(1,0,0,1,1)}+$
$2\,u11'\,u42'\,u51'\,v1'\,v4'\,v5'\,p4^{(0,1)}\,p1^{(1,0)}\,p5^{(1,0)}\,p0^{(1,0,0,1,1)}+$
$2\,u12'\,u41'\,u51'\,v1'\,v4'\,v5'\,p1^{(0,1)}\,p4^{(1,0)}\,p5^{(1,0)}\,p0^{(1,0,0,1,1)}+$
$2\,u11'\,u41'\,u42'\,v1'\,(v4')^2\,p4^{(0,1)}\,p1^{(1,0)}\,p4^{(1,0)}\,p0^{(1,0,0,2,0)}+u12'\,(u41')^2\,v1'$
$(v4')^2\,p1^{(0,1)}\,(p4^{(1,0)})^2\,p0^{(1,0,0,2,0)}+u32'\,v1'\,v3'\,u11''\,p3^{(0,1)}\,p1^{(1,0)}\,p0^{(1,0,1,0,0)}+$
$(u11')^2\,u32'\,v3'\,v1''\,p3^{(0,1)}\,(p1^{(1,0)})^2\,p0^{(1,0,1,0,0)}+u12'\,v1'\,v3'\,u31''\,p1^{(0,1)}\,p3^{(1,0)}\,p0^{(1,0,1,0,0)}+$
$2\,u11'\,u12'\,u31'\,v3'\,v1''\,p1^{(0,1)}\,p1^{(1,0)}\,p3^{(1,0)}\,p0^{(1,0,1,0,0)}+$
$2\,u11'\,u31'\,u32'\,v1'\,v3''\,p3^{(0,1)}\,p1^{(1,0)}\,p3^{(1,0)}\,p0^{(1,0,1,0,0)}+$
$u12'\,(u31')^2\,v1'\,v3''\,p1^{(0,1)}\,(p3^{(1,0)})^2\,p0^{(1,0,1,0,0)}+$
$2\,u11'\,u12'\,u31'\,v1'\,v3'\,p3^{(1,0)}\,p1^{(1,1)}\,p0^{(1,0,1,0,0)}+$
$2\,u11'\,u31'\,u32'\,v1'\,v3'\,p1^{(1,0)}\,p3^{(1,1)}\,p0^{(1,0,1,0,0)}+$
$(u11')^2\,u32'\,v1'\,v3'\,p3^{(0,1)}\,p1^{(2,0)}\,p0^{(1,0,1,0,0)}+u12'\,(u31')^2\,v1'\,v3'\,p1^{(0,1)}\,p3^{(2,0)}\,p0^{(1,0,1,0,0)}+$
$2\,u11'\,u31'\,u52'\,v1'\,v3'\,v5'\,p5^{(0,1)}\,p1^{(1,0)}\,p3^{(1,0)}\,p0^{(1,0,1,0,1)}+$
$2\,u11'\,u32'\,u51'\,v1'\,v3'\,v5'\,p3^{(0,1)}\,p1^{(1,0)}\,p5^{(1,0)}\,p0^{(1,0,1,0,1)}+$
$2\,u12'\,u31'\,u51'\,v1'\,v3'\,v5'\,p1^{(0,1)}\,p3^{(1,0)}\,p5^{(1,0)}\,p0^{(1,0,1,0,1)}+$
$2\,u11'\,u31'\,u42'\,v1'\,v3'\,v4'\,p4^{(0,1)}\,p1^{(1,0)}\,p3^{(1,0)}\,p0^{(1,0,1,1,0)}+$
$2\,u11'\,u32'\,u41'\,v1'\,v3'\,v4'\,p3^{(0,1)}\,p1^{(1,0)}\,p4^{(1,0)}\,p0^{(1,0,1,1,0)}+$
$2\,u12'\,u31'\,u41'\,v1'\,v3'\,v4'\,p1^{(0,1)}\,p3^{(1,0)}\,p4^{(1,0)}\,p0^{(1,0,1,1,0)}+$
$2\,u11'\,u31'\,u32'\,v1'\,(v3')^2\,p3^{(0,1)}\,p1^{(1,0)}\,p3^{(1,0)}\,p0^{(1,0,2,0,0)}+u12'\,(u31')^2\,v1'$
$(v3')^2\,p1^{(0,1)}\,(p3^{(1,0)})^2\,p0^{(1,0,2,0,0)}+u22'\,v1'\,v2'\,u11''\,p2^{(0,1)}\,p1^{(1,0)}\,p0^{(1,1,0,0,0)}+$
$(u11')^2\,u22'\,v2'\,v1''\,p2^{(0,1)}\,(p1^{(1,0)})^2\,p0^{(1,1,0,0,0)}+u12'\,v1'\,v2'\,u21''\,p1^{(0,1)}\,p2^{(1,0)}\,p0^{(1,1,0,0,0)}+$
$2\,u11'\,u12'\,u21'\,v2'\,v1''\,p1^{(0,1)}\,p1^{(1,0)}\,p2^{(1,0)}\,p0^{(1,1,0,0,0)}+$
$2\,u11'\,u21'\,u22'\,v1'\,v2''\,p2^{(0,1)}\,p1^{(1,0)}\,p2^{(1,0)}\,p0^{(1,1,0,0,0)}+$
$u12'\,(u21')^2\,v1'\,v2''\,p1^{(0,1)}\,(p2^{(1,0)})^2\,p0^{(1,1,0,0,0)}+$
$2\,u11'\,u12'\,u21'\,v1'\,v2'\,p2^{(1,0)}\,p1^{(1,1)}\,p0^{(1,1,0,0,0)}+$
$2\,u11'\,u21'\,u22'\,v1'\,v2'\,p1^{(1,0)}\,p2^{(1,1)}\,p0^{(1,1,0,0,0)}+$
$(u11')^2\,u22'\,v1'\,v2'\,p2^{(0,1)}\,p1^{(2,0)}\,p0^{(1,1,0,0,0)}+u12'\,(u21')^2\,v1'\,v2'\,p1^{(0,1)}\,p2^{(2,0)}\,p0^{(1,1,0,0,0)}+$
$2\,u11'\,u21'\,u52'\,v1'\,v2'\,v5'\,p5^{(0,1)}\,p1^{(1,0)}\,p2^{(1,0)}\,p0^{(1,1,0,0,1)}+$
$2\,u11'\,u22'\,u51'\,v1'\,v2'\,v5'\,p2^{(0,1)}\,p1^{(1,0)}\,p5^{(1,0)}\,p0^{(1,1,0,0,1)}+$
$2\,u12'\,u21'\,u51'\,v1'\,v2'\,v5'\,p1^{(0,1)}\,p2^{(1,0)}\,p5^{(1,0)}\,p0^{(1,1,0,0,1)}+$
$2\,u11'\,u21'\,u42'\,v1'\,v2'\,v4'\,p4^{(0,1)}\,p1^{(1,0)}\,p2^{(1,0)}\,p0^{(1,1,0,1,0)}+$
$2\,u11'\,u22'\,u41'\,v1'\,v2'\,v4'\,p2^{(0,1)}\,p1^{(1,0)}\,p4^{(1,0)}\,p0^{(1,1,0,1,0)}+$
$2\,u12'\,u21'\,u41'\,v1'\,v2'\,v4'\,p1^{(0,1)}\,p2^{(1,0)}\,p4^{(1,0)}\,p0^{(1,1,0,1,0)}+$
$2\,u11'\,u21'\,u32'\,v1'\,v2'\,v3'\,p3^{(0,1)}\,p1^{(1,0)}\,p2^{(1,0)}\,p0^{(1,1,1,0,0)}+$
$2\,u11'\,u22'\,u31'\,v1'\,v2'\,v3'\,p2^{(0,1)}\,p1^{(1,0)}\,p3^{(1,0)}\,p0^{(1,1,1,0,0)}+$
$2\,u12'\,u21'\,u31'\,v1'\,v2'\,v3'\,p1^{(0,1)}\,p2^{(1,0)}\,p3^{(1,0)}\,p0^{(1,1,1,0,0)}+$
$2\,u11'\,u21'\,u22'\,v1'\,(v2')^2\,p2^{(0,1)}\,p1^{(1,0)}\,p2^{(1,0)}\,p0^{(1,2,0,0,0)}+u12'\,(u21')^2\,v1'$
$(v2')^2\,p1^{(0,1)}\,(p2^{(1,0)})^2\,p0^{(1,2,0,0,0)}+u12'\,(v1')^2\,u11''\,p1^{(0,1)}\,p1^{(1,0)}\,p0^{(2,0,0,0,0)}+$
$3\,(u11')^2\,u12'\,v1'\,v1''\,p1^{(0,1)}\,(p1^{(1,0)})^2\,p0^{(2,0,0,0,0)}+$
$2\,(u11')^2\,u12'\,(v1')^2\,p1^{(1,0)}\,p1^{(1,1)}\,p0^{(2,0,0,0,0)}+(u11')^2\,u12'\,(v1')^2\,p1^{(0,1)}\,p1^{(2,0)}\,p0^{(2,0,0,0,0)}+$
$(u11')^2\,u52'\,(v1')^2\,v5'\,p5^{(0,1)}\,(p1^{(1,0)})^2\,p0^{(2,0,0,0,1)}+$
$2\,u11'\,u12'\,u51'\,(v1')^2\,v5'\,p1^{(0,1)}\,p1^{(1,0)}\,p5^{(1,0)}\,p0^{(2,0,0,0,1)}+$
$(u11')^2\,u42'\,(v1')^2\,v4'\,p4^{(0,1)}\,(p1^{(1,0)})^2\,p0^{(2,0,0,1,0)}+$
$2\,u11'\,u12'\,u41'\,(v1')^2\,v4'\,p1^{(0,1)}\,p1^{(1,0)}\,p4^{(1,0)}\,p0^{(2,0,0,1,0)}+$
$(u11')^2\,u32'\,(v1')^2\,v3'\,p3^{(0,1)}\,(p1^{(1,0)})^2\,p0^{(2,0,1,0,0)}+$
$2\,u11'\,u12'\,u31'\,(v1')^2\,v3'\,p1^{(0,1)}\,p1^{(1,0)}\,p3^{(1,0)}\,p0^{(2,0,1,0,0)}+$
$(u11')^2\,u22'\,(v1')^2\,v2'\,p2^{(0,1)}\,(p1^{(1,0)})^2\,p0^{(2,1,0,0,0)}+$
$2\,u11'\,u12'\,u21'\,(v1')^2\,v2'\,p1^{(0,1)}\,p1^{(1,0)}\,p2^{(1,0)}\,p0^{(2,1,0,0,0)}+$
$(u11')^2\,u12'\,(v1')^3\,p1^{(0,1)}\,(p1^{(1,0)})^2\,p0^{(3,0,0,0,0)}$

Conclusions

scalar trees

- calibrations
- reduction to bwb trees
- normal forms
- identification via 4-jets

branched models

- vector bw tree, 1-jets
- vector bwb tree, 1-jets
- Kolmogorov graph, N-jets