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Non-Abelian Duality and Confinement in Supersymmetric Gauge Theories

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1 Motivation for SUSY Gauge Theory

Supersymmetry is

- One of the Principles beyond Standard Model.
- Important Basis for Superstring Theory (or Quantum Gravity ?).
- Powerful and Useful Tool for Non-Perturbative Analysis.

Today, We Focus on $\mathcal{N} = 2$ Supersymmetric Gauge Theory, i.e.,

Seiberg-Witten Theory $\sim$ "2-Dim. Ising Model" for Gauge Theory ?

- Possible to Obtain Exact Analytic Solutions $\Longrightarrow$ GOOD "Model"
- Open Up New Field (Duality, CFT, Integrability, Mathematics...)

★ We Try to Extract the Non-Perturbative Information of Gauge Theory (Confinement, Chiral Sym. Breaking) from the Seiberg-Witten Theory.

2 Exact Solutions for $\mathcal{N}=2$ SUSY Gauge Theories

★ $\mathcal{N}=2\ SU(N_c)$ Gauge Theory with N_f Fundamental Hypermultiplets*

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\left[\tau_{cl}\int d^4\theta\,\Phi^\dagger e^V\Phi + \int d^2\theta\,\frac{1}{2}W_\alpha W^\alpha\right] + \mathcal{L}^{(\mathrm{quark})},$$

$$\mathcal{L}^{(\mathrm{quark})} = \sum_i\left[\int d^4\theta\,\left(Q_i^\dagger e^V Q_i + \tilde{Q}_i e^{-V}\tilde{Q}_i^\dagger\right) + \int d^2\theta\,\left(\sqrt{2}\tilde{Q}_i\Phi Q^i + m_i\,\tilde{Q}_i\,Q^i\right) + h.c.\right]$$

and $\mathcal{N}=1$ Soft-Breaking Term :

$$\Delta\mathcal{L} = \int d^2\theta\,\mu\,\mathrm{Tr}\,\Phi^2 + h.c.,$$

where

$$\tau_{cl} \equiv \frac{\theta}{\pi} + \frac{8\pi i}{g^2}, \qquad m_i\ :\ \text{Quark Bare Mass.}$$

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* Φ : Adjoint Rep. and $Q_i,\ \tilde{Q}^i$: Fundamental Rep. ($i=1,\cdots N_f$)

- Seiberg-Witten Exact Solution for $SU(2)$ Pure Yang-Mills Theory (without Q and $\tilde{Q}$)

  At Generic Points on Moduli Space $\langle\Phi\rangle = a\,\frac{\sigma_3}{2} \neq 0$,

$$\text{Gauge Sym}: \quad SU(2) \quad\Longrightarrow\quad U(1).$$

  Low-Energy Effective Action for $U(1)$ Theory :

$$\mathcal{L}_{\text{eff}} \;=\; \frac{1}{4\pi}\,\text{Im}\left[\int d^4\theta A_D A^\dagger + \tau_{\text{eff}}\int d^2\theta W_\alpha W^\alpha\right],$$

where
$$A_D = \frac{\partial\mathcal{F}(A)}{\partial A}, \quad \tau_{\text{eff}} = \frac{\partial^2\mathcal{F}(A)}{\partial A^2}.$$

★ Holomorphic Prepotential $\mathcal{F}(A)$ Completely Determines LEEA

  This LEEA is upto 2-Derivative and "Assumes" Manifest $\mathcal{N}=2$ SUSY in Wilsonian Sense.

◊ BPS States in Low-Energy Effective Theory,

$$M_{BPS} = \sqrt{2}\,|n_e a + n_m a_D|\,,$$

  where n_e and n_m are Electric and Magnetic Charges.

- Seiberg-Witten's Idea

1. LEEA is Invariant under the E-M Duality Transf. $(W, A; \tau) \longrightarrow (W_D, A_D; -\frac{1}{\tau})^{\dagger}$.
2. Classical Moduli Space should be Deformed :

   Dynamical Abelianization $SU(2) \Rightarrow U(1)$ Occurs Everywhere !

3. Quantum Singularity in LEEA is Realized by "Massless" Monopole (or Dyon).

◇ Exact Solution $\sim$ Seiberg-Witten Elliptic Curve :

$$y^2 = x^2(x-u) - \frac{1}{4}\Lambda^4 x \qquad (u = \langle \mathrm{Tr}\, \Phi^2 \rangle).$$

$(a,\ a_D)$ is Given by the SW 1-Form on this Curve :

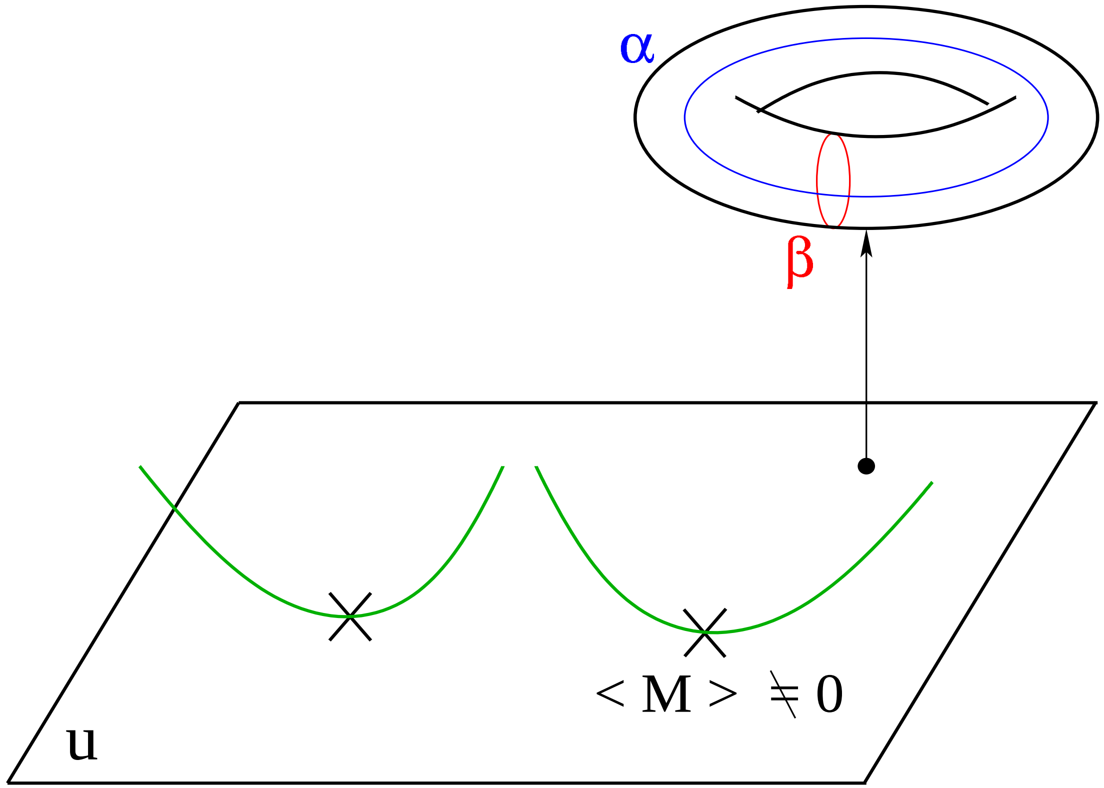
$$a_D(u) = \oint_\beta \lambda_{SW}, \quad a(u) = \oint_\alpha \lambda_{SW}.$$

and $\tau_{\mathrm{eff}}(u) = \partial a_D / \partial a$ is Moduli Parameter of the Torus.

★ Prepotential $\mathcal{F}(A)$ is Obtained from the Soln. $(a,\ a_D)$.

† This can be Shown by a SUSY Legendre Transformation.

- Quantum Moduli Space of $SU(2)$ Pure Yang-Mills Theory



- Confinement from Monopole Condensation

  Breaking to $\mathcal{N} = 1$ by $\Delta W = \mu \, \mathrm{Tr}\, \Phi^2 \Longrightarrow$ Moduli Space is Almost Lifted.

  Effective Superpotential Becomes

$$W_{\text{eff}} = \sqrt{2}\, A_D M \tilde{M} + \mu U(A_D) \qquad (M, \tilde{M} : \text{Monopole Multiplet}).$$

SUSY Vacuum : $a_D = 0$ (Singularity), $\langle M \rangle \propto \sqrt{\mu \frac{\partial u}{\partial a_D}}\Big|_{a_D=0} \neq 0.$

★ Monopole Condensation is Realized $\Longrightarrow$ Color Confinement and Mass Gap.

- Inclusion of “Quarks”

  Important Difference is Existence of Monopoles with Flavor Charge (Jackiw-Rebbi)

$$|\text{Mon.}\rangle, \quad \psi_0^i |\text{Mon.}\rangle, \quad \psi_0^i \psi_0^j |\text{Mon.}\rangle, \cdots \quad (\psi_0^i : \text{Fermionic Zero-Modes})$$

★ Monopole Condensation $\Longrightarrow$ Confinement and Chiral Symmetry Breaking !

E.g., in Massless $N_f = 2$ Case,

$$SO(4) \sim SU(2) \times SU(2) \Longrightarrow SU(2) \times U(1)$$

- Generalizations to $SU(N_c)$ Gauge Theory

  In $SU(N_c)$ Pure Yang-Mills Theory,

  Complete Dynamical Abelianization : $SU(N_c) \rightarrow U(1)^{N_c-1}$

  $\Delta W \implies N_c$ SUSY Vacua with Abelian Monopole Condensation.

- $SU(N_c)$ QCD with N_f Fundamental Quarks

  SW-Curve for Exact Solution of Coulomb Branch :

$$y^2 = \prod_{k=1}^{N_c} (x - a_k)^2 - 4\Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} (x + m_i)$$

$\mathcal{N} = 1$ Vacua (with ΔW) by Minimizing Effective Superpotential,

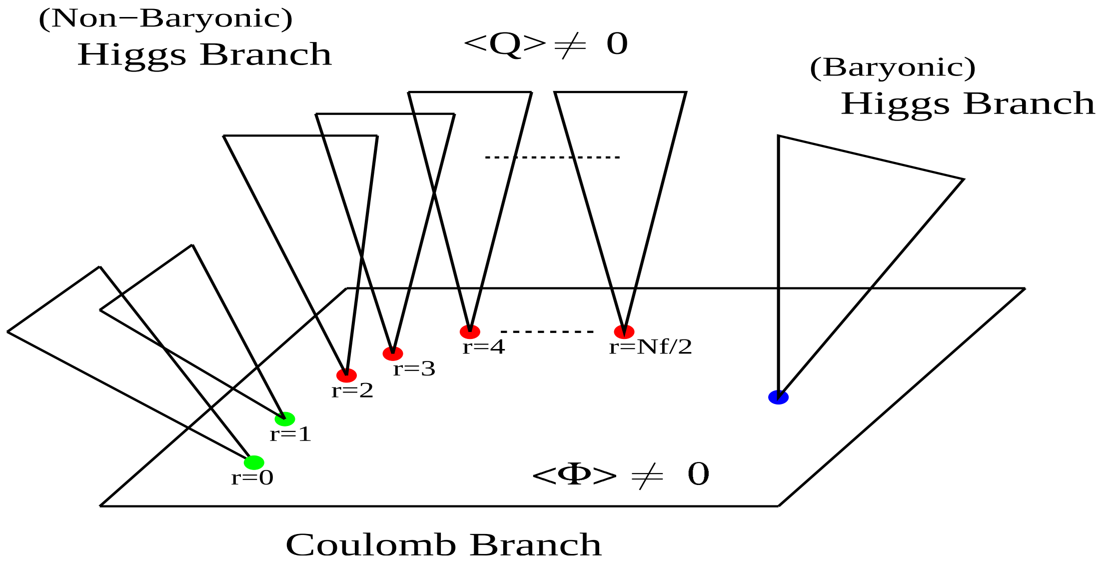
$$W_{\text{eff}} = \sum_{i=1}^{N_c-1} \left(\sqrt{2} A_D^i \tilde{M}^i M_i + S^k m_k \tilde{M}^i M_i \right) + \mu U(A_D^i),$$

$\implies (N_c - 1)$ Pairs of Zero-Points Should Appear in SW-Curve.

In the Equal Mass Case, Higgs Branch, $\langle Q_i \rangle \neq 0$, Also Exists.

Note : Higgs Branch is NOT Modified Quantum Mechanically due to Hyper-Kähler Str.

● Quantum Moduli Space of $\mathcal{N}=2\ SU(N_c)$ QCD with N_f Fund. Quarks ($^{\forall}m_i = m$)



◊ $\mathcal{N}=1$ Vacua with $\Delta W = \mathrm{Tr}\,\mu\,\Phi^2$ ($m=0$)

1. ● : Abelian Vacua with Abelian Monopole Condensation.

2. ● : r-Vacua with $SU(r)\times U(1)^{N_c-r}$ Gauge Sym.

   ∼ Confinement and DSB : $U(N_f) \longrightarrow U(N_f-r)\times U(r)$.

3. ● : Baryonic Vacua with $SU(N_f-N_c)\times U(1)^{2N_c-N_f}$ Gauge Sym.

   ∼ NO Confinement and NO DSB.

● r-Vacua and Baryonic Vacua Has Magnetic DOFs with Charge of $SU(r)\times U(1)^{N_c-r}$.

★ Non-Abelian Magnetic Monopoles ?

Situation is Simlar to "Dual Quarks" in Seiberg Duality[‡].

● Brief Summary of Seiberg Duality

In the Range of $N_c+2\leq N_f<3N_c$,

$\mathcal{N}=1\ SU(N_c)$ SQCD with N_f Quarks
$\Updownarrow$ IR Equivalent
$\mathcal{N}=1\ SU(N_f-N_c)$ SQCD with N_f Quarks and a Singlet Meson

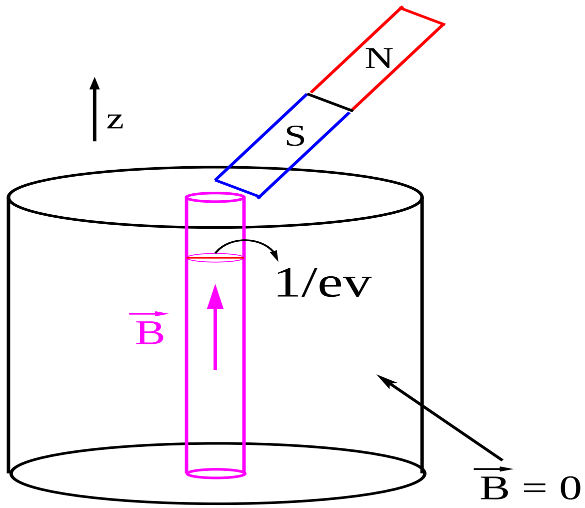
- Seiberg's Conjecture on Existence of the SAME IR Fixed Points.
- Non-Trivial Matching of 't Hooft Anomaly
- Correspondence between Vacuum Str. and Gauge Inv. Chiral Operators.

What is Dual (or Magnetic) Quarks in the Dual Theory ? (in Original Sense)

[‡] Actually, Baryonic Vacua Has been Studied from the Viewpoint of Seiberg Duality.

3 What Can We Learn about Confinement from SW-Theory

★ Vortex Soln. in Abelian Higgs Model as Squeezed Magnetic Flux in Superconductor



- Flux Energy is Proportional to Length

  $\Longrightarrow$ "Probe Monopoles" are Confined.

- Stability from Non Simply-Connected Vacuum Manifold

  $$\Longleftarrow \quad \pi_1(S^1) = \pi_1(U(1)) = \mathbf{Z}.$$

● Electric-Magnetic Duality

$\mathbf{E}$	$\Longleftrightarrow$	$\mathbf{B}$
e	$\Longleftrightarrow$	$g = 1/e$

Abelian Monopole Condensation $\Longrightarrow$ DUAL Meissner Effect.

◇ Confining String $\Longleftrightarrow$ Vortex Soln. in Dual Theory

- However, Abelian Dual Theory Has Different Dynamical Properties from QCD.

$\Longrightarrow$ Spectrum of Vortex Tension Reflects Dynamical Properties.

1. Abelian Theory Has an Infinite Number of STABLE Vortices $\Leftarrow$ $\pi_1(U(1)) = \mathbf{Z}$.
2. Richer Hadron Spectrum from Vortex Spectrum :
   Meson $Q\bar{Q}$ Splits to N_c Mesons due to $SU(N_c) \longrightarrow U(1)^{N_c-1}$.

In Pure $SU(N)$ Yang-Mills Theory (or Heavy Quark Limit[§]), ONLY k-Strings are Stable.

k-String : N-ality k Flux Tube in the Center Z_N of $SU(N)$.

◇ Also, QCD would NOT Have an Effective Weak-Coupling Gauge Theory Description.

Note : Solitonic Vortex in Weakly-Coupled Theory does not Give Linear Regge Trajectory.

★ (Strongly-Coupled) Non-Abelian Effective Description and
Non-Abelian Version of E-M Duality

[§] Actually, Confining Strings in QCD is Unstable due to Production of Quark Pair.

4 Strongly-Coupled SCFT at Argyres-Douglas Fixed Point

Non-Trivial $\mathcal{N}=2$ IR SCFT is Realized at Special Pt. on Moduli Sp. (Argyres-Douglas)

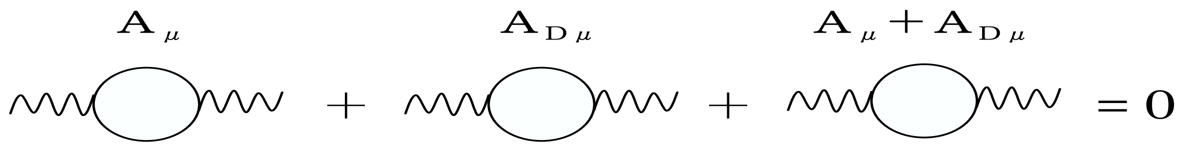
$y^2 = (x + m^*)^3 \implies$ SQED with MASSLESS Mutually Non-Local Dyons

Def. : $n_e^{(1)} \cdot n_m^{(2)} - n_e^{(2)} \cdot n_m^{(1)} \neq 0 \qquad \left(\vec{Q}^{(i)} = (n_e^{(i)}, n_m^{(i)})\right)$

Evidences for Non-Trivial SCFT

1. $\tau_{\text{eff}}(\sim \mathcal{O}(1))$ at the Point is Indep. of Scale in LEET.
2. Scaling Dim. of Chiral Operators Become Fractional.
3. Dynamical Electric and Magnetic Currents Must Coexist $(\Leftarrow$ Conformal Algebra)

Non-Local Cancellation of β-Fn. is Proposed :



$$\sum_i \left(n_e^{(i)} + n_m^{(i)} \tau^{*2} \right) = 0$$

- "Wilsonian" Renormalization Group Approach to AD-Point (Kubota-Yokoi)

M_0 : UV Cut-Off and M : RG Scale (or IR Cut-Off),

Define,
$$\beta\left(\frac{u}{\Lambda^2}, \frac{m_i}{\Lambda}\right) = M\frac{\partial}{\partial M}\tau_{\text{eff}}\left(\frac{u}{\Lambda^2}, \frac{m_i}{\Lambda}\right) = \left(\gamma_u\frac{\partial}{\partial u} + \gamma_{m_i}\frac{\partial}{\partial m_i}\right)\tau_{\text{eff}}\left(\frac{u}{\Lambda^2}, \frac{m_i}{\Lambda}\right).$$

γ_u and γ_{m_i} is Scaling Dimensions : $\gamma_u/\Lambda^2 \equiv M\partial/\partial M\,(u/\Lambda^2)$, etc.

However, We Do NOT Know the M-Dependence (What is IR Cut-Off ?)

Paramtetrize the Flow along $t = M_0/M$ NEAR AD Point ($SU(2),\ N_f = 1$) :

$$\frac{m - m^*}{\Lambda^2} = D_1 t^\alpha, \quad \frac{u - u^*}{\Lambda^2} = D_2 t^\alpha + D_3 t^\beta + \cdots .$$

$$\tau_{\text{eff}} \longrightarrow \tau^* \sim \mathcal{O}(1) \implies \alpha = \tfrac{4}{5},\ \beta = \tfrac{6}{5} \quad (D_1 = D_2).$$

Scaling Dimensions are $[m] = 4/5$, $[u] = 6/5$, $[\tau] = 2/5$.

- Non-Abelian Argyres-Douglas Point (Auzzi-Grena-Konishi, Marmorini-Konishi-Yokoi)

Simplest Example in $SU(3)$ QCD with $N_f = 4$ Flavor

SW-Curve : $$y^2 = \left(x^3 - ux - v\right)^2 - 4\Lambda^2\,(x+m)^4 \,.$$

At $(u,\ v) = (3m^2, 2m^3)$, Curve Becomes

$y^2 \propto (x+m)^4 \implies$ Unbroken $SU(2)$ Symmetry ($r = 2$ Vacuum)

For $m \gg \Lambda$, LEET Becomes $SU(2)$ QCD with 4-Flavor (SCFT !)

For Small m ($\ll \Lambda$), Non-Abelian Generalization of AD-Point Appears !

- Mutually Non-Local DOUBLETS Appear !
- τ^* is $\mathcal{O}(1)$ Fixed Value.

Particle	Charge : $(n_m^1, n_m^2; n_e^1, n_e^2)$
M	$(\pm 1, 1; 0, 0) \times 4$
D	$(\pm 2, -2; \pm 1, 0)$
E	$(0, 2; \pm 1, 0)$

DSB with ΔW : $U(4)_V \to U(2) \times U(2)$

$$\epsilon^{\alpha\beta}\langle M^i_\alpha M^j_\beta \rangle \neq 0$$

Strongly-Coupled Monopole Condensation !

- Another Interesting Example in $USp(4)$ QCD with Massless $N_f = 4$ Flavor

SW-Curve : $$y^2 = x\left(x^2 - ux - v\right)^2 - 4\Lambda^4 x^3.$$

At the Chebyshev Vacua $(u = \pm 2\Lambda^2, v = 0)$, Curve Becomes $y^2 \propto x^4$.

LEET is Also $SU(2) \times U(1)$ Non-Local Theory !

- Extra Doublet C Appears.
- DSB with ΔW : $SO(8) \to U(4)$

$$\delta^{\alpha\beta}\langle M^i_\alpha \tilde{M}^j_\beta\rangle \neq 0.$$

Another Type of Condensation.

Particle	Charge : $(n^1_m, n^2_m; n^1_e, n^2_e)$
M	$(\pm 1, 1; 0, 0) \times 4$
D	$(\pm 2, -2; \pm 1, 0)$
E	$(0, 2; \pm 1, 0)$
C	$(\pm 2, 0; \pm 1, 0$

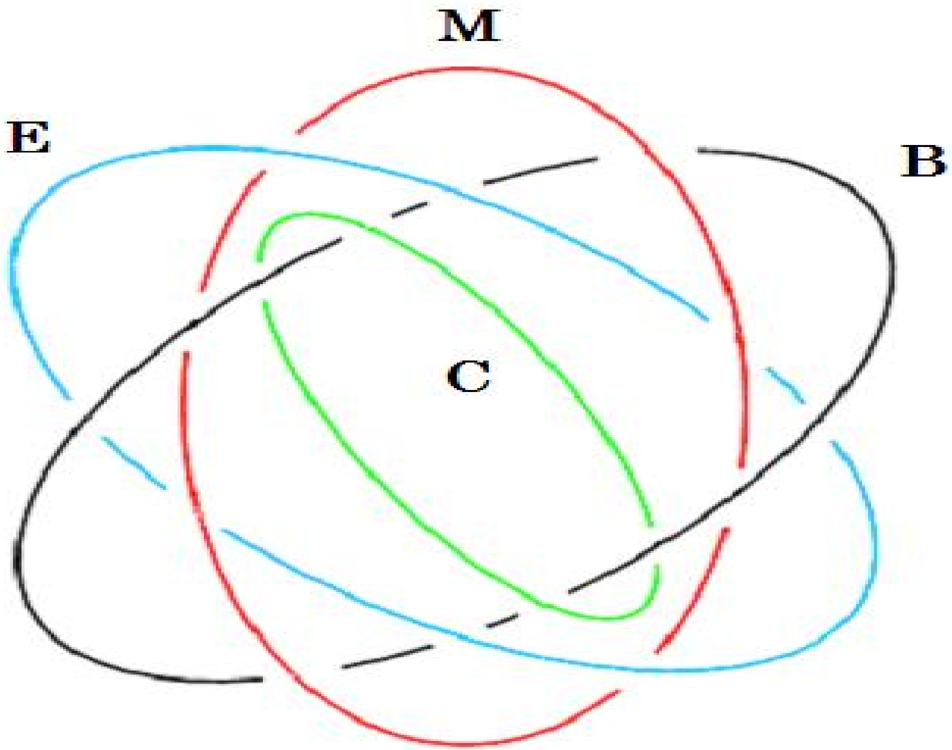
In $USp(2N_c)$ Case, DSB Pattern is $SO(2N_f) \to U(N_f)$.

Non-SUSY Massless $USp(2N_c)$ QCD : $SU(2N_f) \to USp(2N_f)$.

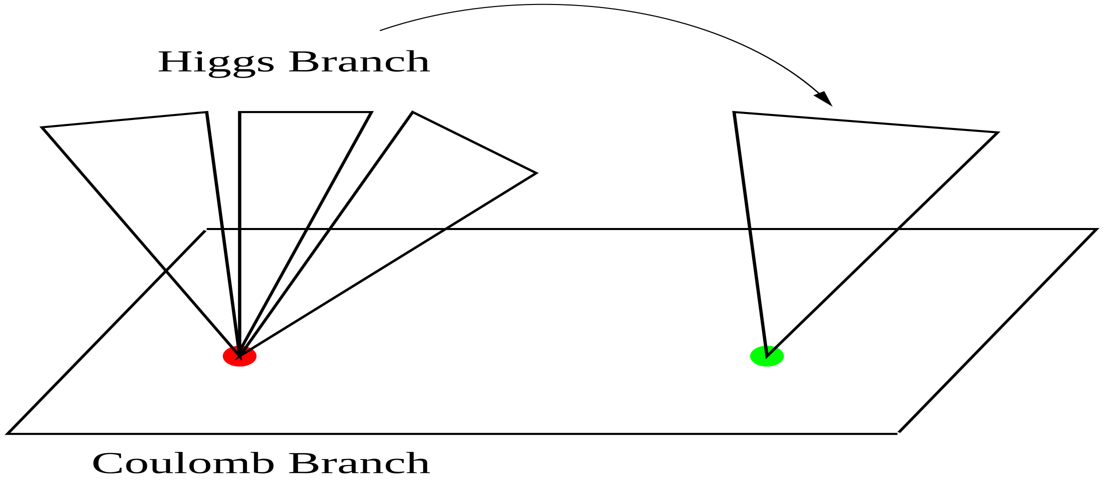
Actually, $USp(2N_f) \cap SO(2N_f) = U(N_f)$.

Hints for Chiral Sym. Breaking in Non-SUSY QCD ?

- Singular Loci in Moduli Space of $USp(4)$ QCD



- Quantum Moduli Space of $USp(2N_c)$ QCD with Massless N_f Flavors



$\mathcal{N} = 1$ SUSY Vacua with ΔW

1. Chebyshev Vacuuum $\Longrightarrow$ Strongly-Coupled Non-Local Eff. Theory

   Dynamical Symmetry Breaking : $SO(2N_f) \Rightarrow U(N_f)$.

2. Baryonic Vacuum $\Longrightarrow$ $USp(2\tilde{N}_c) \times U(1)^{N_c - \tilde{N}_c}$ Gauge Theory¶

   NO Confinement and NO DSB.

¶ $\tilde{N}_c = N_f - N_c - 2$.

5 Introduction to Non-Abelian Duality

Goddard-Nuyts-Olive-Weinberg (GNOW) Duality

For System with the Breaking Pattern, $G \Longrightarrow H$ (H : Non-Abelian) ,

GNOW Duality:

$$H \Longleftrightarrow H^*$$
$$\alpha \Longleftrightarrow \alpha^* = \frac{\alpha}{\alpha \cdot \alpha}$$

★ H^* : DUAL Group Generated by DUAL Root α^*

Example :

$SU(N)$	$\Leftrightarrow$	$SU(N)/Z_N$
$SO(2N)$	$\Leftrightarrow$	$SO(2N)$
$SO(2N+1)$	$\Leftrightarrow$	$USp(2N)$

Note : $U(N)$ is Self-Dual.

- Evidence for GNOW Duality : Non-Abelian Monopoles
  - Topological Argument :

$\pi_2\left(G/H\right)$ is Non-Trivial $\Longrightarrow$ Regular Solitonic Monopoles.

Asymptotic Behavior of Solution at $r \sim \infty$ ($U \in G$, $T_i \in$ C.S.A. of H)

$$\phi \sim U\langle\phi\rangle U^{-1}, \quad F_{ij} \sim \epsilon_{ijk}\frac{x^k}{r^3}\left(\beta \cdot T\right).$$

◇ Generalized Dirac Quantization Condition :

$$2\alpha \cdot \beta \in \mathbb{Z} \qquad \text{for Roots } \alpha \text{ of } H.$$

β Gives a Weight Vector of H^* $\Longrightarrow$ Monopoles Form a Multiplet of H^*

We Discuss the Dual Transformation among these Non-Abelian Monopoles.

In Fact, This is NOT an Easy Task as You See...

6 Brief Summary on Non-Abelian Monopole and Vortex

- (Semi-)Classical Solution for Non-Abelian Monopole

Simple Example : $SU(3)$ Yang-Mills Theory with Ajoint Higgs Φ.

$$SU(3) \overset{\langle\Phi\rangle}{\Longrightarrow} \frac{SU(2)\times U(1)}{Z_2} \quad \text{by} \quad \langle\Phi\rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}$$

In This Case, $\pi_2(G/H) \sim \pi_1\left(\frac{SU(2)\times U(1)}{Z_2}\right) = \mathbb{Z}$.

Regular BPS Solitonic Solution :

$$\begin{aligned} \Phi(x) &= \begin{pmatrix} -\frac{1}{2}v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2}v \end{pmatrix} + 3v\,\vec{S}\cdot\hat{r}\,\phi(r) \\ \vec{A}(x) &= \vec{S}\times\hat{r}\,A(r), \end{aligned}$$

where $\phi(r),\ A(r)$ are BPS-'t Hooft's Profile Function.

- $\vec{S}$ is a Minimal Embedded $SU(2)$ Algebra $(\sigma_a/2)$ in $(1,3)$ (and $(2,3)$) Subspace.

◊ Two Degenerate Solutions $\Rightarrow$ Doublet of Dual $SU(2)$?

In Fact, These Two are Continuously Connected by Unbroken $SU(2)$ Transformation.

Multiplicity of the Monopoles are 1 or 2 or ∞ ?

In Order to Answer the Question, Need to Understand the Tranformation Properties.

However, Some Difficulties are Well-Known in Semi-Classical Analysis for the Solutions

- Non-Normalizable Zero-Modes Appear due to Unbroken $SU(2)$.
- There exists Topological Obstacle to Definition of Charge of the $SU(2)$.

Standard Quantization Procedure Breaks Down due to the Difficulties.

How can We Overcome These Situations ?

$\diamondsuit$ Our Idea : Consider the System with Hierarchical Symmetry Breaking

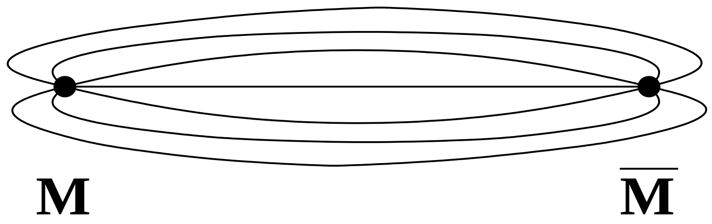
$$G \overset{v_1}{\Longrightarrow} H \overset{v_2}{\Longrightarrow} \emptyset, \qquad v_1 \gg v_2.$$

In this System with $\pi_2(G/H) \neq 0$, Everything Goes Better.

1. At High Energy ($\sim v_1$), $G \longrightarrow H$ Breaking Produces Non-Abelian Monopoles.

2. At Low Energy ($\sim v_2$), Breaking of H Produces Non-Abelian Vortices.

Non-Abelian Monopoles are Confined by Non-Abelian Vortex !



- Low Energy H-Theory is in Higgs Phase $\Rightarrow$ DUAL Theory is in Confining Phase.

  (Cf. H^* is in Higgs Phase $\Rightarrow$ NO Multiplet Structure)

- Light Higgs in the Fundamental Rep. is Needed for Breaking of H.

  $\Longrightarrow$ Massless “Flavor” is Crucial for Non-Abelian Duality (See Later)

- Set Bare Mass Parameter for Quarks $^{\forall}m_i = m \quad (i = 1, 2, \cdots, N_f)$.
  - The r-Vacuum with $r = N$:

$$\Phi = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \dots & m & 0 \\ 0 & \dots & 0 & -N\,m \end{pmatrix},$$

$$Q = \tilde{Q}^{\dagger} = \begin{pmatrix} d & 0 & 0 & 0 & \dots \\ 0 & \ddots & 0 & \vdots & \dots \\ 0 & 0 & d & 0 & \dots \\ 0 & \dots & 0 & 0 & \dots \end{pmatrix}, \quad d = \sqrt{(N+1)\,\mu\, m}.$$

◊ For $\mu \ll m$ (i.e. $d \ll m$),

- Φ Breaks $SU(N+1) \Rightarrow \frac{SU(N)\times U(1)}{Z_N}$ at $v_1 \sim m$
- Q Breaks $\frac{SU(N)\times U(1)}{Z_N}$ Completely at $v_2 \sim d$.

★ However, Diagonal $SU(N)_{C+F} \subset SU(N) \times SU(N_f)$ Sym. is Preserved.

★ High-Energy Theory ($v_2 \longrightarrow 0$) Has (Almost) BPS Monopole Solutions:

$$B_k^A = -\left(\mathcal{D}_k \Phi\right)^A , \qquad B_k^A = \frac{1}{2}\epsilon_{ijk} F_{ij}^A$$

- Mass of Monopoles : $M_{\text{mon}} \sim \frac{2\pi v_1}{g}$.
- Topological Charge from $\pi_2\left(\frac{SU(N+1)}{SU(N)\times U(1)}\right) = \mathbb{Z}$.

★ Low Energy Effective Theory at $(v_2 \lesssim)\ E \ll v_1$

$\mathcal{N} = 2\ SU(N) \times U(1)$ Gauge Theory with N_f Fund. Quarks and "FI-Term".

- The Effective Theory Has (Almost) BPS Vortex Solutions with Tension $\sim 2\pi v_2$.
- Topological Charge from $\pi_1(SU(N) \times U(1)) = \mathbb{Z}$.

◊ These Solutions are BPS Non-Abelian Vortex Solutions.

● Short Review of Non-Abelian Vortex (Hanany-Tong, Auzzi-Bolognesi-Evslin-Konishi-Yung)

Consider the $U(N)$ Gauge Theory

$$\mathcal{L} = \mathrm{Tr}\Big[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - \frac{2}{g^2}\mathcal{D}_\mu\Phi^\dagger\mathcal{D}^\mu\Phi - \mathcal{D}_\mu H\,\mathcal{D}^\mu H^\dagger - \lambda\left(c\,\mathbf{1}_N - H\,H^\dagger\right)^2\Big] + \mathrm{Tr}\,[\,(H^\dagger\Phi - m\,H^\dagger)(\Phi\,H - m\,H)\,],$$

where Φ : Adjoint Higgs, H : $N\ (= N_f)$ Fundamental Higgs in Matrix Form.

The Vacuum of this Theory:

$$\langle\Phi\rangle = m\,\mathbf{1}_N, \qquad \langle H\rangle = \sqrt{c}\,\mathbf{1}_N.$$

★ The Vacuum Preserves Color-Flavor Diagonal Sym. $SU(N)_{C+F}$.

● Eq. of Motion for BPS Case $\left(\lambda = g^2/4\right)$:

$$(\mathcal{D}_1 + i\mathcal{D}_2)\ H = 0, \quad F_{12} + \frac{g^2}{2}\left(c\,\mathbf{1}_N - H\,H^\dagger\right) = 0.$$

◇ "Non-Abelian" Zero Modes from the Breaking of $SU(N)_{C+F}$ by Vortex.

◇ Moduli Matrix Formalism for Non-Abelian Vortex (Eto-Isozumi-Nitta-Ohashi-Sakai)

Solutions for the Eq. of Motion ($z = x_1 + ix_2$):

$$H = S^{-1}(z,\bar{z})\, H_0(z), \quad A_1 + i\, A_2 = -2\, i\, S^{-1}\, \bar{\partial}_z S(z,\bar{z}).$$

- $S(z,\bar{z})$ Satisfies a Nonlinear “Master Equation”:

$$\partial_z\, (\Omega^{-1}\partial_{\bar{z}}\, \Omega) = \frac{g^2}{4}\, (c\, 1_N - \Omega^{-1}\, H_0\, H_0^\dagger). \quad (\Omega \equiv SS^\dagger)$$

- $H_0(z)$ is Moduli Matrix Encoding All Moduli Parameters up to the V-Transformation :

$$H_0(z) \to V(z)H_0(z), \quad S(z,\bar{z}) \to V(z)S(z,\bar{z}) \quad (V \text{ is any Hol. Matrix}).$$

● Another Construction of Moduli Space of k-vortex by Kähler Quotient

$$\frac{\{H_0(z) |\, \det H_0 \sim z^k\}}{\{V(z) |\, V \in GL(N_c; \mathrm{C})\}} \Longleftrightarrow \frac{\{\mathrm{Z}, \Psi |\, (k \times k) \text{ and } (N_c \times k) \text{ Const. Matrix}\}}{\{U |\, U \in GL(k; \mathrm{C})\}},$$

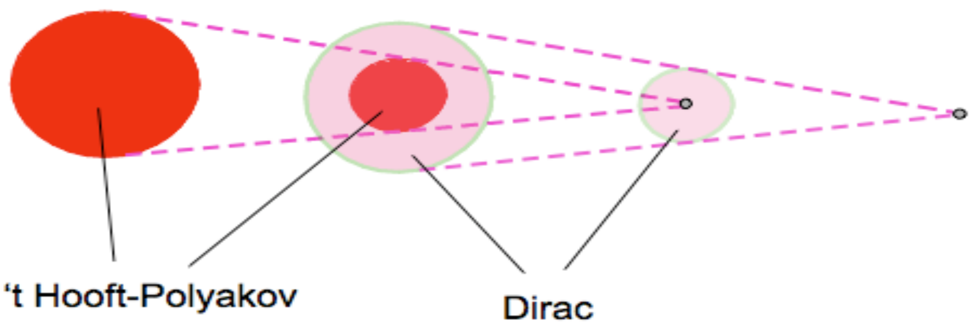
where $Z \sim UZU^{-1}$ and $\Psi \sim \Psi U^{-1}$.

- 1-Vortex for $U(N)$ Theory : $\mathcal{M}_{k=1} = \mathbb{C}P^{N-1}$.
- Composite 2-Vortex in $U(2)$ Theory : $\mathcal{M}_{k=2} = W\mathbb{C}P^2_{(2,1,1)}$.

7 Non-Abelian Duality from Monopole-Vortex Complex

◊ Monopole-Vortex Complex from Topological Argument, “Exact Homotopy Sequence”:

$$\ldots \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \ldots$$



In Our Full Theory, $\pi_2(G) = 0$ ($\Longrightarrow$ NO Stable Monopole) and

$\pi_1(G) = 0$ ($\Longrightarrow$ NO Stable Vortex).

$$\pi_2\left(\frac{SU(N+1)}{U(N)}\right) = \pi_2(CP^N) \sim \pi_1(U(N)) = \mathbb{Z}$$

High-Energy Monopole $\Longleftrightarrow$ Low-Energy Vortex

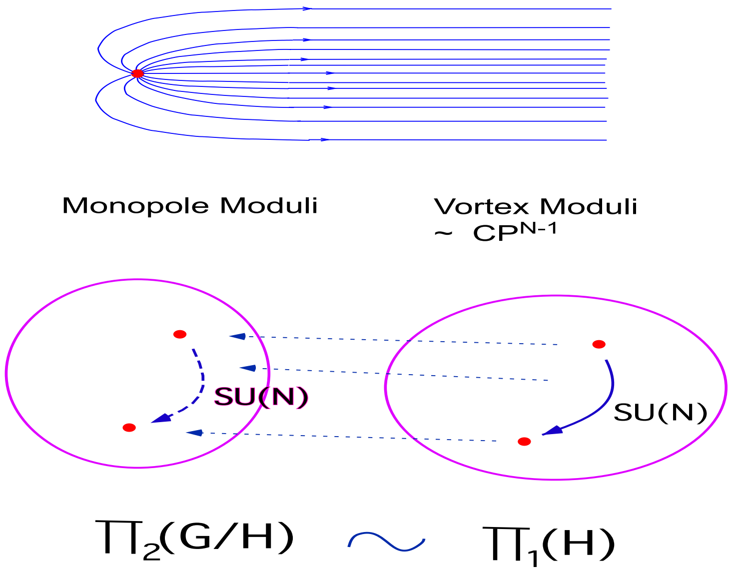
★ Monopoles should be Confined by Vortices !

◇ Dual Transformations among Monopoles from Vortex (Eto et. al.)

A Vortex Solution Breaks Color-Flavor Diagonal Sym.

$$SU(N)_{C+F} \longrightarrow SU(N-1) \times U(1)$$

- Moduli Space for 1-Vortex : $\mathcal{M} = SU(N)/U(N-1) = \mathbb{C}P^{N-1}$.



★ We can Show the Moduli Parameters Transform as N-Rep. under $SU(N)_{C+F}$.

$\Longrightarrow$ High-Energy Non-Abelian Monopoles Form an N-Rep Multiplet.

- Simplest Example for $SU(2)\times U(1)$ Theory

Moduli Matrix up to V-Transformation

$$H_0^{(1,0)} \simeq \begin{pmatrix} z-z_0 & 0 \\ -b_0 & 1 \end{pmatrix}, \quad H_0^{(0,1)} \simeq \begin{pmatrix} 1 & -a_0 \\ 0 & z-z_0 \end{pmatrix}.$$

- a_0 and b_0 are Orientational Moduli and Correspond to Two Patches of $\mathbb{C}P^1$.
- Under $SU(2)_{C+F}$ Transformation :

$$H_0 \to V(z)\, H_0\, U^\dagger, \quad U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad (|\alpha|^2+|\beta|^2=1),$$

Moduli Parameter a_0 Transforms as

$$a_0 \to \frac{\alpha\, a_0 + \beta}{\alpha^* - \beta^*\, a_0}.$$

★ This is Nothing But the Transformation of Doublet.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad a_0 \equiv \frac{a_1}{a_2}.$$

◇ This Derivation Does NOT Rely on Semi-Classical Analysis of Monopole

- General $SU(N) \times U(1)$ Case

A Standard Form of Moduli Matrix for Minimal 1-Vortex:

$$H_0(z) \simeq \begin{pmatrix} 1 & 0 & 0 & -a_1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -a_{N-1} \\ 0 & \dots & 0 & z - z_0 \end{pmatrix}.$$

Under the $SU(N)_{C+F}$ Transformation and V-Transformation,

$$H(z, \bar{z}) \rightarrow UHU^\dagger \quad \Longrightarrow \quad H_0(z) \rightarrow V(z) H_0 U^\dagger.$$

$\Diamond$ For $U = 1 + X$, a_i Transform as Inhomogenious Coordinates of $\mathbb{C}P^{N-1}$.

Note: Homogenious Coord. of $\mathbb{C}P^{N-1}$ Transform as N Rep. under $SU(N)$ Isometry.

- Our Result is Consistent with Quantum Result from Seiberg-Witten Solution

With Appropriate Number of Flavors, on the Quantum $r = N$ Vacuum,

Effective Theory Has $SU(N) \times U(1)$ Gauge Sym. with Monopoles of N Rep.

★ Another Non-Trivial Example : $SO(2N+1) \to U(N) \to \emptyset$

- Simplest Case for $SO(5) \to U(2) \to \emptyset$.

  Essential Differences : $\pi_1(SO(5)) = Z_2$

  - Minimal Monopole is Dirac-Type and Minimal Vortex is Truly Stable.

(1). Vortex Side : We have Investigated Moduli Space of Composite 2-Vortex (See Next)

$$\mathcal{M}_{k=2} = W\mathbb{C}P^2_{(2,1,1)} \simeq \mathbb{C}P^2/Z_2.$$

- Bulk of $W\mathbb{C}P^2$: **Triplet** under $SU(2)_{C+F}$.
- Conical Singularity : **Singlet**.

(2). Monopole Side : Regular Solutions with 1–Parameter Not Related to Sym. (E. Weinberg)

Fortunately, Moduli Space and Metric is KNOWN in This Case,

$$\mathcal{M}_{\text{mon}} = \mathbb{C}^2/Z_2 \simeq H_0^{(1,1)} \quad : \quad \text{A Patch of } W\mathbb{C}P^2$$

- A "Compactification" of $\mathcal{M}_{\text{mon}}$ Gives $W\mathbb{C}P^2$.

★ Monopoles Transform : $\mathbf{3} \oplus \mathbf{1} (= \mathbf{2} \otimes \mathbf{2})$.

- Moduli Matrix for Composite 2-Vortex in $U(2)$ Theory ($z_0 = 0$)

$$H_0^{(0,2)}(z) = \begin{pmatrix} 1 & -az - b \\ 0 & z^2 \end{pmatrix}, \quad H_0^{(1,1)}(z) = \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z - \phi \end{pmatrix},$$

with the Constraint $\phi^2 + \eta\tilde{\eta} = 0$.

The Constraint in $H_0^{(1,1)}$ can be Solved as

$$XY = -\phi, \quad X^2 = \eta, \quad Y^2 = -\tilde{\eta}.$$

◊ Correct Coord. of Moduli Space : $(X, Y) \sim (-X, -Y) \Longrightarrow \mathbb{C}P^2/Z_2$.

- Another Representation of $H_0^{(1,1)}$

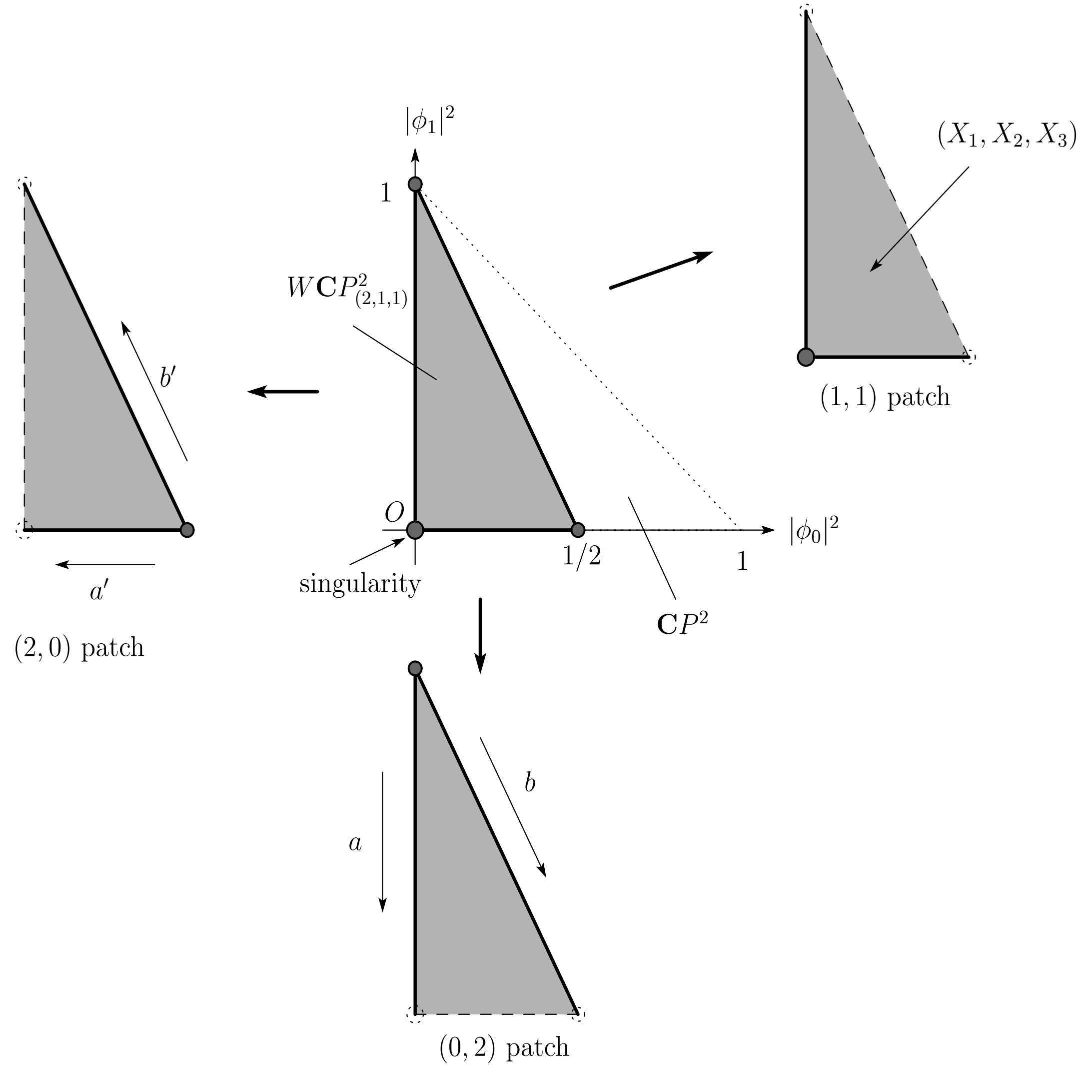
$$H_0^{(1,1)}(z) = z\mathbf{1}_2 + \vec{X} \cdot \vec{\sigma} \qquad (X_3 = \phi, \overset{(\sim)}{\eta} = X_1 \mp X_2)$$

Under $SU(2)_{C+F}$ Transf. $H_0^{(1,1)}(z) \rightarrow U H_0 U^\dagger$ (with $V(z) = U$)

$$\Longrightarrow \vec{X} \text{ Transforms as a Triplet of } SU(2)_{C+F}.$$

Note : A Point $\vec{X} = 0$ is Invariant $\Longrightarrow$ Nothing But the Singularity $X = Y = 0$.

- Picture of Moduli Space of Composite 2-Vortex in $U(2)$ Theory



● Non-Abelian Dual Symmetry as Color-Flavor Diagonal Symmetry

- Color-Flavor Diagonal Sym. $SU(N)_{C+F}$ is EXACT Symmetry of the Theory.

  $\Longrightarrow$ Energy of Whole Monopole-Vortex Complex is Invariant.

- In High Energy Theory $(v_2 \longrightarrow 0)$, This Sym. Acts as ONLY Color Part of $SU(N)_{C+F}$.

  $\Longrightarrow$ In Full Theory, This Sym. Becomes Non-Local Sym. Involving Flavor !

  ★ Dual Transformation as Non-Local Transformation by $SU(N)_{C+F}$

★ Quantum Aspects of Non-Abelian Duality

In Full-Quantum Theory, This Dual Sym. $SU(N)_{C+F}$ Has Trouble.

- According to Famous Seiberg-Witten Results,

  Strong Coupling Dynamics Breaks $SU(N)$ to ABELIAN $U(1)^{N-1}$.

  To Resolve this, $N_f \geq 2N$ Massless Flavors are Crucial

  $\Longrightarrow$ Low-Energy Theory Becomes Infra-Red Free Due to Flavors.

Note : EXACT Flavor Sym. (Equal Mass) is Needed for Non-Abelian Dual Theory.

8 Semilocal Non-Abelian Vortex

- Abrikosov-Nielsen-Olesen (ANO) Vortex in Abelian Higgs Model

  Finite Tension Soln. of Eq. of Motion : For BPS Case,

$$(\mathcal{D}_x + i\mathcal{D}_y)\,\Phi = 0, \qquad B + \frac{1}{2}\left(|\Phi|^2 - v^2\right) = 0.$$

  - Stability from Non Simply-Connected Vacuum Manifold $\Longleftarrow \quad \pi_1(S^1) = \mathbf{Z}$.
  - Characterized by Position Moduli on the Plane.

- What Happens for Multi-Flavor Case, e.g. for $N_f = 2$ Case ?

  1. Vacuum Manifold Changes to S^3 : $\pi_1(S^3) = 1$ (Trivial) $\Longrightarrow$ Stable Solution ?

     ◊ For $\lambda/e^2 \leq 1$, Stable Solutions Do Exist and Classified by $\pi_1(U(1))$.

  2. For $\lambda/e^2 = 1$, Vortex Solutions Have Transverse “Size Moduli” other than Positions !
  3. Large r Behavior is Quite Different from ANO $\Rightarrow$ “Lump” in Sigma Models.

  These Vortex Solutions are Called Semilocal Vortex (or String) (Vachaspati-Achucarro).

- Moduli Space for Semilocal Non-Abelian Vortex with $N_f > N_c$ (Eto et. al.)

  - Non-Trivial Degenerate Higgs Vacua Appear:

$$\mathcal{V}_{\text{Higgs}} \simeq \frac{SU(N_f)}{SU(N_c) \times SU(N_f - Nc) \times U(1)}$$

$\Longrightarrow$ $SU(N_c)_{C+F} \times SU(N_f - N_c)$ Global Symmetry is Preserved.

  - Moduli Matrix Becomes Rectangular : $H_0(z) = (D(z), Q(z))$,
    where $D(z)$: $N_c \times N_c$ Matrix and $Q(z)$: $N_c \times (N_f - N_c)$ Matrix.

$\Longrightarrow$ Additional "Size" Moduli Appear from $Q(z)$.

  - Vortex Number k $\Longleftarrow$ $\det H_0 H_0^\dagger \sim |z|^{2k} \qquad (|z| \sim \infty)$.

However, Kähler Quotient Construction Can be Also Applied to Semilocal Case !

- Construction of Moduli Space by Kähler Quotient

$$\frac{\{H_0^{(k)}(z)\}}{\{V(z)\}} \Longleftrightarrow \frac{\{\mathbf{Z}, \Psi, \widetilde{\Psi} \,|\, (k \times k), (N_c \times k), (k \times (N_c - N_f)) \text{ Matrix}\}}{\{U \,|\, U \in GL(k; \mathrm{C})\}},$$

where $GL(k;\mathrm{C})$ Action : $\{Z, \Psi, \widetilde{\Psi}\} \sim \{UZU^{-1}, \Psi U^{-1}, U\widetilde{\Psi}\}$,

and U is Free on $\{Z, \Psi\}$: $\{UZU^{-1}, \Psi U^{-1}\} = \{Z, \Psi\} \Longrightarrow U = 1.$

- Simplest Example : 1-Vortex in $U(2)$ Theory with $N_f = 3$ ($GL(1;\mathrm{C}) = \mathrm{C}^*$)

$$\left(\mathbf{Z}, \Psi, \widetilde{\Psi}\right) \sim \left(\mathbf{Z}, \lambda^{-1}\Psi, \lambda\widetilde{\Psi}\right), \qquad \lambda \in \mathrm{C}^*,$$

where Z, $\widetilde{\Psi}$: Constant and Ψ : 2-Vector.

- Except for Position Moduli Z, Moduli Space Appears to be

$$W\mathbb{C}P^2[1, 1, -1] \; : \; (y_1, y_2, y_3) \sim (\lambda\, y_1, \lambda\, y_2, \lambda^{-1}\, y_3) \quad (\neq (0, 0, 0)).$$

★ This Space is NON-Hausdorff Space !

Because Two Distinct Points $(a, b, 0)$ and $(0, 0, 1)$ Have NO Disjoint Neighborhoods.‖

‖ $(\epsilon\, a, \epsilon\, b, 1) \sim (a, b, \epsilon)$, where ϵ is Arbitrarily Small.

- Two "Regularized" Spaces as Moduli Spaces of "Dual" Theories

  In Order to Make the Space Hausdorff, We Should Eliminate Either Point:

★ Two "Regularizations" $\Longrightarrow$ Two Different Manifolds

This Corresponds to the Choice Between $U(2)$ Theory and "Dual" $U(1)$ Theory.

1. $W\mathbb{C}P^2[\underline{1,1},-1] \equiv W\mathbb{C}P^2[1,1,-1] - (0,0,1)$

   Moduli Space of $U(2)$ Theory $\Longrightarrow$ $\mathcal{M}_{2,3} = \widetilde{\mathbf{C}}^2$: Blow Up of $\mathbf{C}^2$

2. $W\mathbb{C}P^2[1,1,\underline{-1}] \equiv W\mathbb{C}P^2[1,1,-1] - \mathbb{C}P^1$

   Moduli Space of "Dual" $U(1)$ Theory $\Longrightarrow$ $\mathcal{M}_{1,3} = \mathbf{C}^2$

★ $GL(k,\mathbf{C})$ Free Condition $\Longleftrightarrow$ Removing "Irregular" Subspace.

- Generalization to $U(N_c)$ with N_f : Parent Space is $W\mathbb{C}P^{N_f-1}[1^{N_c},-1^{N_f-N_c}]$.

1. $\mathcal{M}_{N_c,N_f} = W\mathbb{C}P^{N_f-1}[\underline{1^{N_c}},-1^{\widetilde{N}_c}]\ :\ \mathcal{O}(-1)^{\oplus\widetilde{N}_c} \to \mathbb{C}P^{N_c-1}$,

2. $\mathcal{M}_{\widetilde{N}_c,N_f} = W\mathbb{C}P^{N_f-1}[1^{N_c},\underline{-1^{\widetilde{N}_c}}]\ :\ \mathcal{O}(-1)^{\oplus N_c} \to \mathbb{C}P^{\widetilde{N}_c-1}$,

where
$$\widetilde{N}_c = N_c - N_f.$$

- Lump Solution in Strong Coupling Limit

LEET of Strong Coupling Limit $\Longrightarrow$ Non-Linear Sigma Model on $\mathcal{V}_{\text{Higgs}}$.

This Sigma Model Has Codim. 2 Lump Solitons from $\pi_2(\mathcal{V}_{\text{Higgs}}) = \mathbf{Z}$.

$\Longrightarrow$ In the Strong Coupling Limit, Our Vortex Becomes this Lump Soliton.

Moduli Space of Smooth k-Lump Soliton is Also Determined by Moduli Matrix :

$$\begin{aligned}\mathcal{M}^{\text{lump}}_{N_c,N_f} &= \left\{(\mathbf{Z},\Psi,\widetilde{\Psi})|GL(k,\mathbf{C}) \text{ free on } (\mathbf{Z},\Psi) \text{ and } (\mathbf{Z},\widetilde{\Psi})\right\}/GL(k,\mathbf{C}) \\ &= \mathcal{M}_{N_c,N_f} \cap \mathcal{M}_{\widetilde{N}_c,N_f}.\end{aligned}$$

Finally, We Have the Following Diamond Diagram:

$$\widehat{\mathcal{M}}_{N_c,N_f}\ (W\mathbb{C}P^2_{(1,1,-1)})$$

$$\swarrow \qquad\qquad \searrow$$

$$\mathcal{M}_{N_c,N_f}\ (\widetilde{\mathbf{C}}^2) \longleftarrow\cdots \text{ Seiberg-like Duality } \cdots\longrightarrow \mathcal{M}_{\widetilde{N}_c,N_f}\ (\mathbf{C}^2)$$

$$\underset{g^2\to\infty}{\searrow} \qquad\qquad \underset{\tilde{g}^2\to\infty}{\swarrow}$$

$$\mathcal{M}^{\text{lump}}_{N_c,N_f}\ ((\mathbf{C}^2)^*)$$

9 Worldsheet Effective Action of Moduli on Vortex

- Worldsheet Effective Theory on Vortex

  Possible to Obtain Eff. Theory by Promoting the Moduli to Slowly-Moving Fields

  $\Downarrow$

  2-Dim. Non-Linear Sigma Model on Our Moduli Space

  In SUSY Context, Moduli Matrix Can Also Provide the Kähler Potential :

$$K = \mathrm{Tr} \int d^2 z \left(\xi \log \Omega + \Omega^{-1} H_0 H_0^\dagger + \mathcal{O}(1/g^2) \right).$$

  Note : This Gives Standard $\mathbb{C}P^N$ Metric for Local NA-Vortex.

  Crucial Difference from Local Vortex is Existence of Non-Normalizable Moduli.

- An Example for $U(2)$ Theory with $N_f = 3$ (L : IR Cut-Off)

$$K_{N_c=2,N_f=3} = \xi\pi|c|^2(1+|b|^2)\log\frac{L^2}{|c|^2(1+|b|^2)} + \mathcal{O}(L^0).$$

  Replacement ($\tilde{c} = c,\ \tilde{b} = c\, b$) Gives $K_{N_c=1,N_f=3}$ of $U(1)$ Dual Theory.

10 Z_N Vortex in $\mathcal{N} = 1^* SU(N)$ Gauge Theory

Our $\mathcal{N} = 1^*$ Gauge Theory is

A Mass-Deformed $\mathcal{N} = 4$ Super Yang-Mills Theory.

★ (Deformed Version of) Montonen-Olive (MO) Duality :

Electric Flux : Center of $SU(N) \sim Z_N$

$\Updownarrow$ Dual

Magnetic Flux : $\pi_1 \left(SU(N)/Z_N\right) \sim Z_N$

★ Note : This Duality Exchanges Also Confining Phase for Higgs Phase.

◊ Magnetic Strings can be Constructed as Solitonic Vortex.

$\Longrightarrow$ (In Some Regime,) Semiclassical Analysis can be Applied !

We Discuss the Solitonic Vortex with Charge k of N-ality in $\mathcal{N} = 1^* SU(N)$ Gauge Theory.

- $\mathcal{N} = 1^* \; SU(N)$ Gauge Theory with Φ_I^a ($I = 1, 2, 3$)

  Superpotential

$$W = i\frac{\sqrt{2}}{g^2} f_{abc} \Phi_1^a \, \Phi_2^b \, \Phi_3^c + \frac{m}{2\, g^2} \sum_{I=1}^{3} \Phi_I^{a\,2}.$$

$\Diamond$ Theory has $SO(3)$ Symmetry as a Flavor Symmetry.

- $E \gg m$, Theory is (Approximately) Scale Invariant.
- $E \ll m$, Theory Becomes $\mathcal{N} = 1$ Pure Yang-Mills Theory

SUSY Vacuum Condition** $\Rightarrow \; \Phi_I = i \frac{m}{\sqrt{2}} X_I \; : \; X_I$ is Some Rep. of $SU(2)$.

1. Higgs Vacuum : X_I is N-Dim. Irrep. and Gauge Sym. is Completely Broken.
2. Confining Vacuum : X_I is Trivial Rep. and Gauge Sym. is Unbroken.
3. Coulomb Vacua : X_I is some Smaller Rep. and Gauge Sym. is Partially Broken.

** $\sqrt{2}\, [\Phi_I, \, \Phi_J] + m \, \epsilon_{IJK} \, \Phi_K = 0$ and $\left[\bar{\Phi}_I, \Phi_I\right] = 0$.

- On the Higgs Vaccum,

All $\langle \Phi_I \rangle \propto m \implies SU(N)$ Gauge Sym. is Broken at $E \sim m$.

$$\Downarrow$$

Weak-Coupling Effective Theory below $m \longrightarrow$ Semiclassical Analysis

- Solitonic Z_N Vortex Solution (Marmorini-Konishi-Vinci-Yokoi)

The Center Z_N is Trivial on Adjoint Fields : $SU(N) \overset{\langle \Phi_I \rangle}{\Longrightarrow} Z_N$

◇ Non-Trivial Winding Characterized by $\pi_1\,(SU(N)/Z_N) \sim Z_N$

Ansatz for the Fields : $f(\infty) = \phi_I(\infty) = 1$ and $f(0) = \phi_1(0) = \phi_2(0) = 0$,

- Gauge Fields : $A_\varphi(r) = (f(r)/r)\,\beta\,T_0$, Others $= 0$.
- Matter Fields : $\Phi_I(r) = \phi_I(r)\,V(\varphi)\,\Phi_I^{(0)}\,V^{-1}(\varphi)$,
  $\Phi_I^{(0)} = i(m/\sqrt{2})X_I$.

- Generator of Center

$$T_0 = \frac{1}{N}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & -(N-1) \end{pmatrix}$$

$$V(\varphi) = \exp\left(i\,\beta\,T_0\,\varphi\right), \qquad V(2\pi) \in Z_N \quad (\beta \in \mathbf{Z})$$

- Explicit Solution for $SU(2)$ and $SU(3)$ Vortex $\quad (\phi_1(r) = \phi_2(r) \equiv \phi(r))$

Take $X_I = \sigma_I/2 \ (SU(2))$ and $(X_I)_{a\,b} = -i\,\epsilon_{I\,a\,b} \ (SU(3))$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right) + C_1\,\frac{2(1-f)}{r}\,m^2\,\phi^2 = 0,$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\eta}{dr}\right) + m^2\left(3\,\phi^2 - \eta - 2\,\eta\,\phi^2\right) = 0,$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - C_2\,\beta^2\,\frac{(1-f)^2}{r^2}\phi - m^2\left(\phi^3 + \eta^2\,\phi - 3\,\eta\,\phi + \phi\right) = 0$$

- Non-Abelian Orientational Moduli

The Higgs Vacuum is a Kind of "Color-Flavor Locking" Phase

- Global Flavor Sym. : $X_I \to X'_I = R_{I\,J}\, X_J, \quad R_{I\,J} \in SO(3)$
- Gauge Sym. : $X_I \to X'_I = U\, X_I\, U^\dagger, \quad U \in SU(N)$

For Particular Gauge Transformation $\tilde{U} = \exp\left(i\,\alpha^I X_I\right)$,

$$X_I \to X'_I = \tilde{U}\, X_I\, \tilde{U}^\dagger = R_{I\,J}\, X_J \qquad (\tilde{U} \in SU(2) \subset SU(N))$$

$\Longrightarrow$ SO(3) Color-Flavor Diagonal Symmetry ††

- Worldsheet Effective Theory on the Vortex

- Our Vortex Solution Breaks This $SO(3)_{\mathrm{C+F}}$ Sym. to $U(1)$ $(\phi_1 = \phi_2 \neq \phi_3)$
- Z_N Vortex should NOT be BPS.

$\Longrightarrow$ 2-Dim. Non-SUSY CP^1 $(\sim\ SO(3)/U(1))$ σ-Model

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†† For $SU(2)$ Case, Discussed by Markov-Marshakov-Yung.

11 Worldsheet Dynamics on Vortex and MO-Duality

Interesting Observation by Markov-Marshakov-Yung : On Higgs Vacuum,

Confinement of Monopole $\Longleftrightarrow$ Confinement of Kinks on Vortex

- MO-Duality in $\mathcal{N} = 4$ SYM Implies Dual-Description by Monopole in ADJOINT Rep.
- 2-Dim. Non-SUSY CP^1 σ Model Has Mass-Gap and Triplet-Meson under $SO(3)$.

★ Non-Abelian Duality from Vortex Moduli Dynamics

This is Consistent for $SU(2) \sim SO(3)$ Case $\Longrightarrow$ How about $SU(3)$ Case ?

Also for $SU(3)$ Case, Orientational Moduli Appears to be Same as $CP^1 \sim SO(3)/U(1)$.

★ Can Dual $SU(3)$ Sym. be Understood from this Eff. Theory ?
Or Does Some Additional Moduli Exist ?

Moduli for Composite Vortex might be Useful for This Problem.

12 Summary and Overview

We Have Discussed

- Confinement from Strongly-Coupled Monopole Condensation at NA-AD Point.
- Non-Abelian (GNO) Duality from Moduli of NA-Vortex.
- Seiberg-like Duality from Moduli of Semilocal NA-Vortex.
- Montonen-Olive Duality from Z_N Vortex in $\mathcal{N} = 1^*$ Theory.

● Outlook

- More Detailed Study and Deeper Understanding of All the Idea is Needed.
- Extract More Information about Quark Confinement from Seiberg-Witten Theory.
- Understanding of Dynamics of NA-Vortex and NA-Monopole.
  $\Longrightarrow$ Relation to String and D-Brane Dynamics.

Many Interesting Problems are Remaining in SUSY Gauge Theory !