Realization of symmetry in the ERG approach to quantum field theory

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Abstract

We overview the exact renormalization group (ERG) approach to quantum field theory. In particular we explain how to introduce symmetry, either local or global, in the presence of a finite momentum cutoff.

Plan of the talk

1. Wilson-Polchinski ERG differential equations
2. Perturbation theory
3. Realization of symmetry
4. Perturbative applications
   (a) QED
   (b) YM theories
   (c) WZ model

Summary

1. ERG can be formulated without resort to perturbation theory. (the original intention of K. Wilson)

2. Overview of the perturbative construction of a renormalized theory using ERG.

3. Essential technical tools to introduce symmetry in the ERG approach.

4. Practical applications. (QED, YM, WZ)

5. Most theorists are not familiar with the formalism. We aim to popularize the ERG approach.

6. Please consult with the lecture notes by H.S. `arXiv:0710.1662`

Wilson-Polchinski ERG differential equations

1. Let $S[\phi]$ be the action of a real scalar field theory in D dimensional euclidean space.

2. We generate a one-parameter family of actions S_t equivalent to S:

$$\exp\left[S_t[\phi]\right] = \int [d\phi']\, \mathrm{e}^{S[\phi']}$$
$$\times \exp\left\{ -\frac{1}{2}\int_p A_t(p)^2 \left(\phi(p) - Z_t(p)\phi'(p)\right)\left(\phi(-p) - Z_t(p)\phi'(-p)\right) \right\}$$

   (a) $Z_t(p)$ is the change of field normalization:

$$\phi(p) \sim Z_t(p)\phi'(p)$$

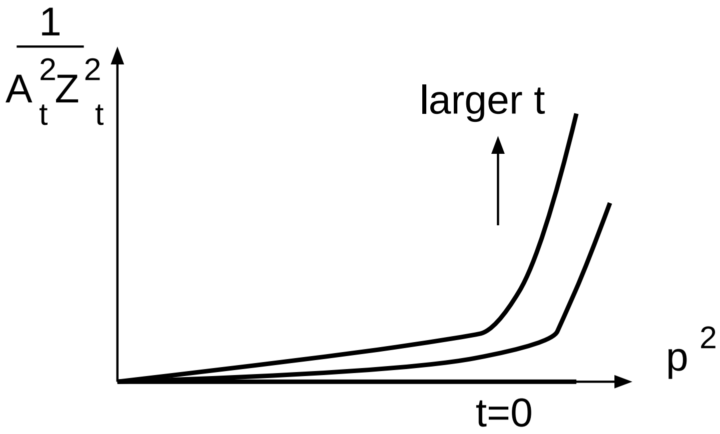
(b) $\frac{1}{A_t(p)}$ is the width of field diffusion:

$$\left|\frac{1}{Z_t(p)}\phi(p) - \phi'(p)\right| \sim \frac{1}{A_t(p)Z_t(p)}$$

i.e., $S_t[\phi]$ is obtained by an incomplete integration of $S[\phi']$.

(c) We take $\frac{1}{A_0} = 0$, $Z_0 = 1$ so that $S_0 = S$.

(d) More integration for larger p^2, larger t.



3. $\underline{S_t \text{ is equivalent to } S}$ in the following sense:

(a) Define the generating functionals:

$$\begin{cases} e^{W[J]} & \equiv \int [d\phi] \exp\left[S[\phi] + i \int_p J(p)\phi(-p)\right] \\ e^{W_t[J]} & \equiv \int [d\phi] \exp\left[S_t[\phi] + i \int_p J(p)\phi(-p)\right] \end{cases}$$

(b) A simple gaussian integration gives

$$e^{W_t[J]} = \exp\left\{-\frac{1}{2}\int_p \frac{1}{A_t(p)^2} J(p)J(-p) + W\left[Z_t(p)J(p)\right]\right\}$$

Hence,

$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_{S_t} & = \frac{1}{A_t(p)^2} + Z_t(p)^2 \,\langle\, \phi(p)\phi(-p)\,\rangle_S \\ \langle\, \phi(p_1)\cdots\phi(p_{n>1})\,\rangle_{S_t}^{\mathrm{c}} & = \prod_{i=1}^n Z_t(p_i)\cdot \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_S^c \end{cases}$$

(c) Conversely,

$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_S & = \frac{1}{Z_t(p)^2}\,\langle\, \phi(p)\phi(-p)\,\rangle_{S_t} - \frac{1}{A_t(p)^2 Z_t(p)^2} \\ \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_S^c & = \prod_{i=1}^n \frac{1}{Z_t(p_i)} \cdot \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_{S_t}^c \end{cases}$$

The original correlation functions can be constructed as long as A_t and Z_t are finite.

Hence, S_t and S are equivalent.

4. The t dependence of the action is given by the ERG differential equation of Wilson (Wilson & Kogut '74, sect. 11):

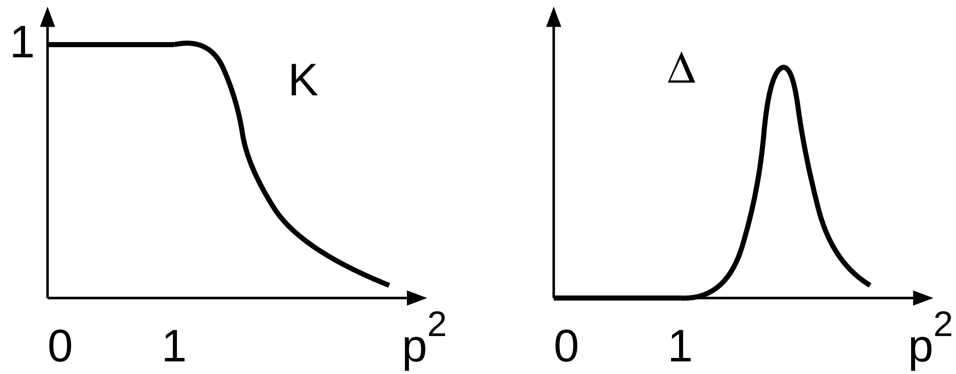
$$\partial_t S_t = \int_p \left[F_t(p) \cdot \phi(p) \frac{\delta S_t}{\delta \phi(p)} + G_t(p) \cdot \frac{1}{2} \left\{ \frac{\delta S_t}{\delta \phi(p)} \frac{\delta S_t}{\delta \phi(-p)} + \frac{\delta^2 S_t}{\delta \phi(p) \delta \phi(-p)} \right\} \right]$$

where

$$\begin{cases} F_t(p) \equiv -\partial_t \ln Z_t(p) \\ G_t(p) \equiv -2 \frac{1}{A_t(p)^2} \partial_t \ln \left(A_t(p) Z_t(p) \right) \end{cases}$$

5. Choice of Z_t:

(a) Let $K(p)$ be an arbitrary cutoff function and $\Delta(p) \equiv -2p^2 dK(p)/dp^2$.

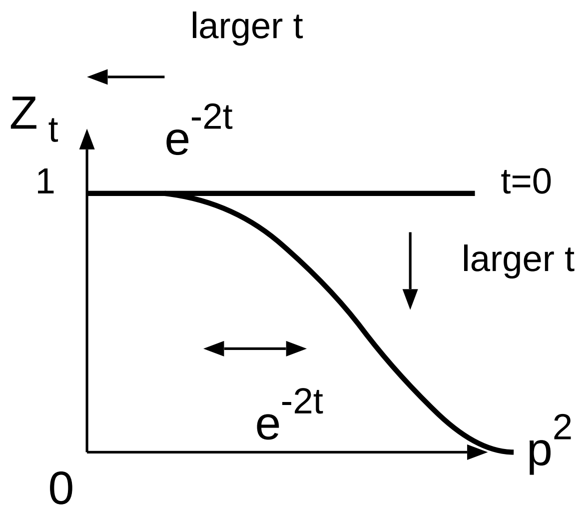


(b) We choose

$$\begin{cases} Z_t(p) &= \frac{K(pe^t)}{K(p)} \exp\left\{\frac{1}{2}\int_0^t dt'\,\eta(t')\right\} \\ F_t(p) &= \frac{\Delta(pe^t)}{K(pe^t)} - \frac{\eta(t)}{2} \end{cases}$$

where η is an anomalous dimension.

(c) For $\eta = 0$, we obtain



This implies S_t has smaller field fluctuations for larger t and larger p^2, since

$$\langle\, \phi(p_1) \cdots \phi(p_n)\, \rangle_{S_t}^c = \prod_{i=1}^n Z_t(p_i) \cdot \langle\, \phi(p_1) \cdots \phi(p_n)\, \rangle_S^c$$

6. Three examples of A_t:

(a) Wilson

$$\begin{cases} \frac{1}{A_t(p)^2} = \mathrm{e}^{2t} - \frac{K(p\mathrm{e}^t)^2}{K(p)^2}\,\mathrm{e}^{\int_0^t \eta} & \stackrel{t\to\infty}{\longrightarrow} \mathrm{e}^{2t} \\ G_t(p) = 2\mathrm{e}^{2t}\left(\frac{\Delta(p\mathrm{e}^t)}{K(p\mathrm{e}^t)} + 1 - \frac{1}{2}\eta(t)\right) & \end{cases}$$

This implies

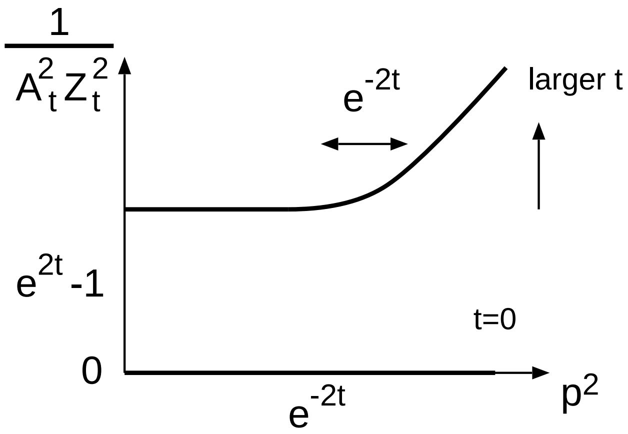
$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_{S_t} - \mathrm{e}^{2t} = \mathrm{e}^{\int_0^t \eta}\,\frac{K(p\mathrm{e}^t)^2}{K(p)^2}\left(\langle\, \phi(p)\phi(-p)\,\rangle_S - 1\right) \\ \langle\, \phi(p_1)\cdots\phi(p_{2n})\,\rangle_{S_t} = \mathrm{e}^{n\int_0^t \eta}\prod_{i=1}^{2n}\frac{K(p_i\mathrm{e}^t)}{K(p_i)}\cdot\langle\, \phi(p_1)\cdots\phi(p_{2n})\,\rangle_S \end{cases}$$

This is the original choice of Wilson. In this choice, the two-point function behaves as

$$\langle\, \phi(p)\phi(-p)\,\rangle_{S_t} \stackrel{t\to\infty}{\longrightarrow} \mathrm{e}^{2t}$$

For $\eta = 0$, the squared width of the incomplete integration is

$$\frac{1}{A_t^2 Z_t^2} = \mathrm{e}^{2t} \frac{K(p)^2}{K(p\mathrm{e}^t)^2} - 1$$



The anomalous dimension $\eta(t)$ is determined by the condition

$$-\frac{\partial}{\partial p^2} \frac{\delta^2 S_t}{\delta\phi(p)\delta\phi(-p)}\Big|_{\phi=0} \stackrel{p^2\to 0}{\longrightarrow} 1$$

(b) Polchinski ($\eta = 0$)

$$\begin{cases} \frac{1}{A_t(p)^2} & = \frac{K(pe^t)}{p^2+m^2}\left(1 - \frac{K(pe^t)}{K(p)}\right) \\ G_t(p) & = \frac{\Delta(pe^t)}{p^2+m^2} \end{cases}$$

This implies

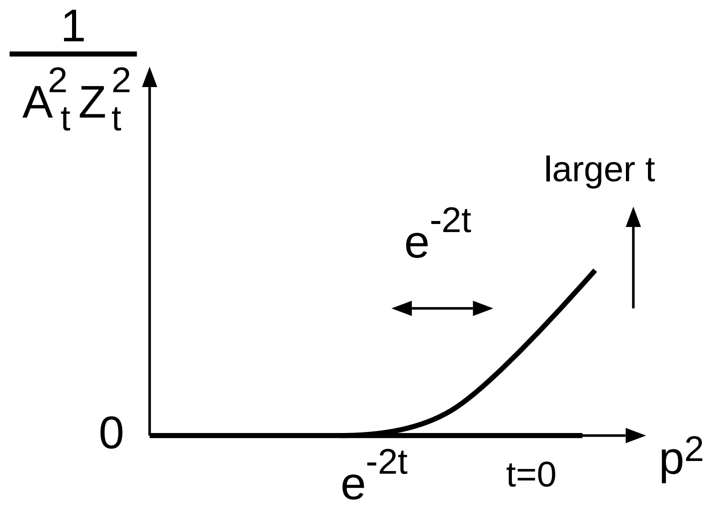
$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_{S_t} - \frac{1}{p^2+m^2}K(pe^t)\left(1 - K(pe^t)\right) \\ \quad = \frac{K(pe^t)^2}{K(p)^2}\left(\langle\, \phi(p)\phi(-p)\,\rangle_S - \frac{1}{p^2+m^2}K(p)(1-K(p))\right) \\ \langle\, \phi(p_1)\cdots\phi(p_{2n})\,\rangle_{S_t} = \prod_{i=1}^{2n}\frac{K(p_i e^t)}{K(p_i)}\cdot\langle\, \phi(p_1)\cdots\phi(p_{2n})\,\rangle_S \end{cases}$$

The two-point function behaves as

$$\langle\, \phi(p)\phi(-p)\,\rangle_{S_t} \xrightarrow{t\gg 1} \frac{K(pe^t)}{p^2+m^2}$$

The squared width of the incomplete integration is

$$\frac{1}{A_t(p)^2 Z_t(p)^2} = \frac{K(p)}{p^2+m^2}\left(\frac{K(p)}{K(p\mathrm{e}^t)} - 1\right)$$



(c) Polchinski ($\eta \neq 0$, running squared mass $m^2(t)$)

$$\left\{ \begin{array}{rcl} \frac{1}{A_t(p)^2} & = & K(p\mathrm{e}^t)^2 \left\{ \frac{1}{p^2+m^2(t)} \left(\frac{1}{K(p\mathrm{e}^t)} - 1 \right) - \frac{\mathrm{e}^{\int_0^t \eta}}{p^2+m^2(0)} \left(\frac{1}{K(p)} - 1 \right) \right\} \\ G_t(p) & = & \frac{1}{p^2+m^2(t)} \left\{ \Delta(p\mathrm{e}^t) - \left(\eta(t) + \frac{\frac{d}{dt} m^2(t)}{p^2+m^2(t)} \right) K(p\mathrm{e}^t) \left(1 - K(p\mathrm{e}^t)\right) \right\} \end{array} \right.$$

This implies

$$\left\{ \begin{array}{c} \langle\, \phi(p)\phi(-p) \,\rangle_{S_t} - \frac{1}{p^2+m^2(t)} K(p\mathrm{e}^t) \left(1 - K(p\mathrm{e}^t)\right) \\ = \mathrm{e}^{\int_0^t \eta} \frac{K(p\mathrm{e}^t)^2}{K(p)^2} \left(\langle\, \phi(p)\phi(-p) \,\rangle_S - \frac{1}{p^2+m^2(0)} K(p)(1 - K(p)) \right) \\ \langle\, \phi(p_1) \cdots \phi(p_{2n}) \,\rangle_{S_t} = \mathrm{e}^{n \int_0^t \eta} \prod_{i=1}^{2n} \frac{K(p_i \mathrm{e}^t)}{K(p_i)} \cdot \langle\, \phi(p_1) \cdots \phi(p_{2n}) \,\rangle_S \end{array} \right.$$

The anomalous dimension $\eta(t)$ and the scale dependence of $m^2(t)$ are determined so that

$$\frac{\delta^2 S_t}{\delta\phi(p)\delta\phi(-p)}\Big|_{\phi=0} = -m^2(t) - p^2 + \cdots$$

for small momenta.

7. **Fixed points** — Rescaling of space is necessary:

$$p' = p\mathrm{e}^t \Longrightarrow \boxed{\phi'(p') \equiv \mathrm{e}^{-\frac{D+2}{2}t}\phi(p)}$$

This modifies the ERG differential equation to

$$\begin{aligned}
\partial_t S_t \quad &= \quad \int_{p'} \Bigg[\left\{ p'_\mu \frac{\partial \phi'(p')}{\partial p'_\mu} + \left(\frac{D+2}{2} + F'_t(p') \right) \phi'(p') \right\} \frac{\delta S_t}{\delta \phi'(p')} \\
&\quad + G'_t(p') \frac{1}{2} \left\{ \frac{\delta S_t}{\delta \phi'(p')} \frac{\delta S_t}{\delta \phi'(-p')} + \frac{\delta^2 S_t}{\delta \phi'(p') \delta \phi'(-p')} \right\} \Bigg]
\end{aligned}$$

where

$$\left\{ \begin{array}{ccc} F'_t(p') \equiv F_t(p'\mathrm{e}^{-t}) & \equiv & \frac{\Delta(p')}{K(p')} - \frac{\eta(t)}{2} \\ G'_t(p') & \equiv & \mathrm{e}^{-2t} G_t(p'\mathrm{e}^{-t}) \end{array} \right.$$

8. **Wilson-Fisher fixed point S^*_{WF} for $D=3$**

(a) Given

$$S[\phi] \equiv -\frac{1}{2}\int_p \frac{1}{K(p)}\frac{1}{2}\phi(-p)\phi(p)\left(p^2+m_0^2\right) - \frac{\lambda_0}{4!}\int \phi^4$$

we construct a one-parameter family S_t with $S_0 = S$.

(b) At criticality $m_0^2 = m_{cr}^2(\lambda_0)$,

$$\lim_{t\to\infty} S_t = S^*_{WF}$$

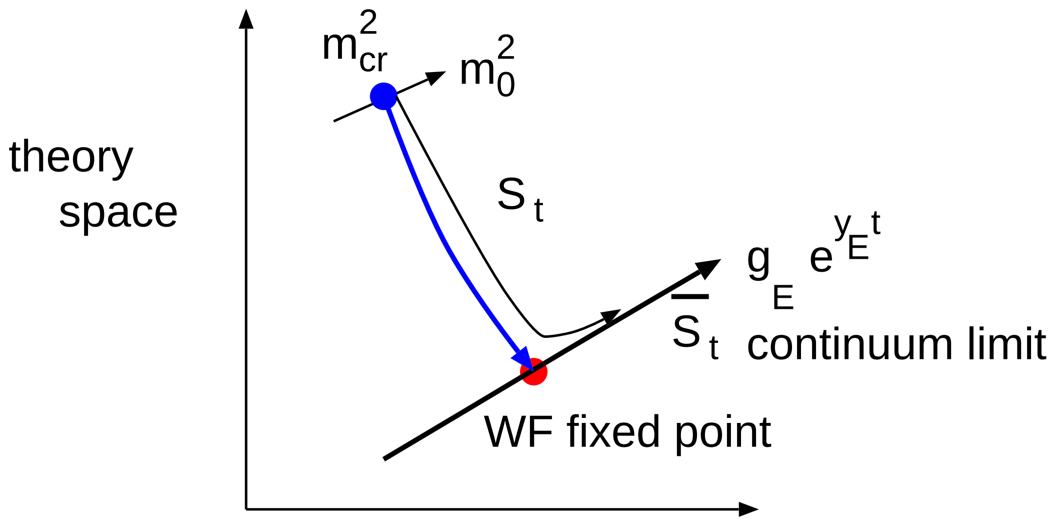
(c) The continuum limit can be constructed as

$$\bar{S}_t \equiv \lim_{s\to\infty} S_s(m_0^2 = m_{cr}^2(\lambda_0) + g_E \mathrm{e}^{y_E(t-s)})$$

where g_E is an arbitrary constant.

(d) $\bar{S}_t$ satisfies the ERG differential equation and the asymptotic condition

$$\lim_{t \to -\infty} \bar{S}_t = S^*_{WF}$$



(e) S^*_{WF} has been constructed only approximately. Numerically in LPA (local potential approx.) by various groups, and perturbatively by H.S. '04

Perturbation theory

1. We split

$$S(\Lambda) = S_{\text{free}}(\Lambda) + S_{\text{int}}(\Lambda)$$

where S_{free} is the free action

$$S_{\text{free}}(\Lambda) \equiv -\frac{1}{2}\int_p \phi(-p)\phi(p)\frac{p^2+m^2}{K\left(\frac{p}{\Lambda}\right)}$$

which gives the free propagator

$$\langle\, \phi(p)\phi(-p)\,\rangle_{S_{\text{free}}(\Lambda)} = \frac{K\left(\frac{p}{\Lambda}\right)}{p^2+m^2}$$

Λ is the **momentum cutoff**.

2. ERG describes how the action changes as we lower the momentum cutoff.

3. The interaction action $S_{\rm int}(t)$ satisfies the Polchinski ERG differential equation (Polchinski '83)

$$\boxed{-\Lambda\frac{\partial}{\partial\Lambda}S_{\rm int}(\Lambda) = \int_p \frac{\Delta(p/\Lambda)}{p^2+m^2}\frac{1}{2}\left\{\frac{\delta S_{\rm int}}{\delta\phi(-p)}\frac{\delta S_{\rm int}}{\delta\phi(p)} + \frac{\delta^2 S_{\rm int}}{\delta\phi(-p)\delta\phi(p)}\right\}}$$

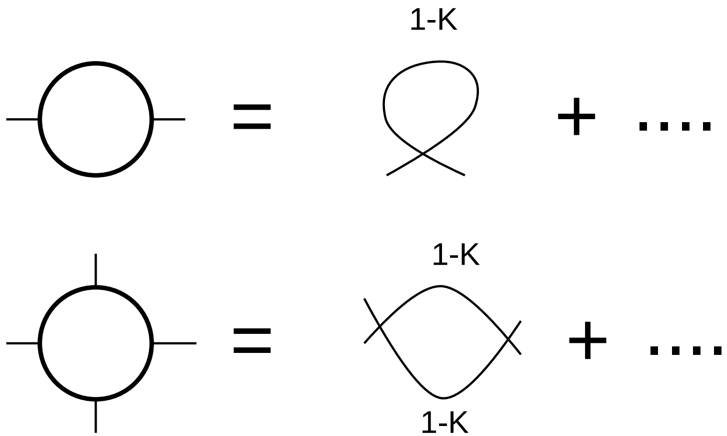
4. The ERG differential equation guarantees the Λ independence of the correlation functions:

$$\begin{cases} \langle\, \phi(p)\phi(-p)\,\rangle_\infty & \equiv\ \frac{1-1/K\left(\frac{p}{\Lambda}\right)}{p^2+m^2} + \frac{1}{K\left(\frac{p}{\Lambda}\right)^2}\langle\, \phi(p)\phi(-p)\,\rangle_{S(\Lambda)} \\ \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_\infty & \equiv\ \prod_{i=1}^n \frac{1}{K\left(\frac{p_i}{\Lambda}\right)}\cdot\langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_{S(\Lambda)} \end{cases}$$

5. Physically, $S_{\rm int}$ consists of elementary vertices and the propagators

$$\frac{1 - K\left(\frac{p}{\Lambda}\right)}{p^2 + m^2}$$

with an IR cutoff at Λ.



$S_{\rm int}$ is defined precisely in terms of the ERG differential equation and the asymptotic conditions given below.

6. Renormalized theories are characterized by the following asymptotic behavior (H.S. '03): (We consider only $D = 4$ from now on.)

$$S_{\text{int}}(\Lambda) \quad \stackrel{\Lambda\to\infty}{\longrightarrow} \quad \int \Bigg[\left(\Lambda^2 a_2(\ln\Lambda/\mu) + m^2 b_2(\ln\Lambda/\mu)\right) \frac{1}{2}\phi^2 + c_2(\ln\Lambda/\mu)\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + a_4(\ln\Lambda/\mu)\frac{1}{4!}\phi^4 \Bigg]$$

where m and the momentum of ϕ is assumed small compared to Λ, and μ is an arbitrary renormalization scale.

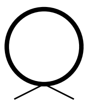
7. The theory has three parameters: ($a_2(0)$ cannot be controlled.)
   (a) $b_2(0)$ normalizes m^2
   (b) $c_2(0)$ normalizes ϕ
   (c) $a_4(0)$ defines the coupling constant λ

(d) The MS scheme

$$\boxed{b_2(0) = c_2(0) = 0, \quad a_4(0) = -\lambda}$$

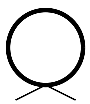
is especially convenient for practical loop calculations.

8. For example, the one-loop two-point vertex is given by



$$= -\frac{\lambda}{2}\left(-\frac{\Lambda^2}{2}\int_q \frac{\Delta(q)}{q^2} + \frac{2}{(4\pi)^2} m^2 \ln\frac{\Lambda}{\mu} + m^4 \int_q \frac{1 - K(q/\Lambda)}{q^4(q^2+m^2)}\right)$$

$$\xrightarrow{\Lambda\to\infty} -\frac{\lambda}{2}\left(-\frac{\Lambda^2}{2}\int_q \frac{\Delta(q)}{q^2} + \frac{2}{(4\pi)^2} m^2 \ln\frac{\Lambda}{\mu}\right)$$

This satisfies $-\Lambda\frac{\partial}{\partial\Lambda}$  $= -\frac{\lambda}{2}\int_p \frac{\Delta(p/\Lambda)}{p^2+m^2}$

9. A **composite operator** $\mathcal{O}(p)$

$$\equiv \sum_n \frac{1}{n!} \int_{p_1,\cdots,p_n} \phi(p_1)\cdots\phi(p_n)\, \mathcal{O}_n(\Lambda; p_1,\cdots,p_n)\, (2\pi)^4 \delta^{(4)}(p_1 + \cdots p_n - p)$$

satisfies the ERG differential equation

$$\boxed{-\Lambda \frac{\partial}{\partial \Lambda} \mathcal{O}(p) = \mathcal{D} \cdot \mathcal{O}(p)}$$

where

$$\mathcal{D} \equiv \int_p \frac{\Delta(p/\Lambda)}{p^2 + m^2} \left(\frac{\delta S_{\mathrm{int}}}{\delta\phi(-p)} \frac{\delta}{\delta\phi(p)} + \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right)$$

With small ϵ, we can regard $S + \epsilon\mathcal{O}$ as a small deformation of the action.

10. The correlation functions

$$\langle\, \mathcal{O}(p)\phi(p_1)\cdots\phi(p_n)\,\rangle_\infty \equiv \prod_{i=1}^n \frac{1}{K\left(\frac{p_i}{\Lambda}\right)} \cdot \langle\, \mathcal{O}(p)\phi(p_1)\cdots\phi(p_n)\,\rangle_S$$

are independent of the cutoff Λ.

11. A simple composite operator:

$$[\phi](p) \equiv \phi(p) + \frac{1-K\left(\frac{p}{\Lambda}\right)}{p^2+m^2}\frac{\delta S_{\rm int}}{\delta\phi(-p)}$$

is the composite operator corresponding to the elementary field $\phi(p)$:

$$\langle\, \phi(p)\phi(p_1)\cdots\phi(p_n)\,\rangle_\infty = \prod_{i=1}^n \frac{1}{K\left(\frac{p_i}{\Lambda}\right)} \cdot \langle\, [\phi](p)\phi(p_1)\cdots\phi(p_n)\,\rangle_S$$

12. The correlation functions do depend on the renormalization scale μ:

$$\left(-\mu\frac{\partial}{\partial\mu} + m^2\beta_m(\lambda)\partial_{m^2} + \beta(\lambda)\partial_\lambda - n\gamma(\lambda)\right)\langle\,\phi(p_1)\cdots\phi(p_n)\,\rangle_\infty = 0$$

This **RG equation** is derived from the operator equality (H.S. '06)

$$\boxed{-\mu\partial_\mu S = m^2\beta_m\mathcal{O}_m + \beta\mathcal{O}_\lambda + \gamma\mathcal{N}}$$

where $\mathcal{O}_m, \mathcal{O}_\lambda, \mathcal{N}$ are composite operators with the properties:

$$\left\{\begin{array}{rcl} \langle\,\mathcal{O}_m\phi(p_1)\cdots\phi(p_{2n})\,\rangle_\infty &=& -\partial_{m^2}\langle\,\phi(p_1)\cdots\phi(p_{2n})\,\rangle_\infty \\ \langle\,\mathcal{O}_\lambda\phi(p_1)\cdots\phi(p_{2n})\,\rangle_\infty &=& -\partial_\lambda\langle\,\phi(p_1)\cdots\phi(p_{2n})\,\rangle_\infty \\ \langle\,\mathcal{N}\phi(p_1)\cdots\phi(p_{2n})\,\rangle_\infty &=& 2n\,\langle\,\phi(p_1)\cdots\phi(p_{2n})\,\rangle_\infty \end{array}\right.$$

These operators are given in terms of the action as

$$\begin{cases} \mathcal{O}_m & \equiv & -\partial_{m^2} S - \int_q \frac{K(q/\Lambda)(1-K(q/\Lambda))}{(q^2+m^2)^2} \frac{1}{2} \left\{ \frac{\delta S}{\delta\phi(q)} \frac{\delta S}{\delta\phi(-q)} + \frac{\delta^2 S}{\delta\phi(q)\delta\phi(-q)} \right\} \\ \mathcal{O}_\lambda & \equiv & -\partial_\lambda S \\ \mathcal{N} & \equiv & -\int_q \phi(q) \frac{\delta S}{\delta\phi(q)} - \int_q \frac{K(q/\Lambda)(1-K(q/\Lambda))}{q^2+m^2} \left\{ \frac{\delta S}{\delta\phi(q)} \frac{\delta S}{\delta\phi(-q)} + \frac{\delta^2 S}{\delta\phi(q)\delta\phi(-q)} \right\} \end{cases}$$

Realization of symmetry

1. A simple observation: $S(\Lambda)$ with a finite Λ gives the correlation functions in the continuum limit.

2. Whatever symmetry of the continuum limit must be realized in $S(\Lambda)$.

3. **Universal form of invariance**

   (a) Given a composite operator $\mathcal{O}(p)$,

$$\Sigma \equiv \int_p K\left(\frac{p}{\Lambda}\right) \mathrm{e}^{-S} \frac{\delta}{\delta\phi(p)} \left(\mathcal{O}(p)\mathrm{e}^{S}\right) = \int_p K\left(\frac{p}{\Lambda}\right) \left(\frac{\delta S}{\delta\phi(p)} \mathcal{O}(p) + \frac{\delta\mathcal{O}(p)}{\delta\phi(p)}\right)$$

   is also a composite operator.

(b) $\int_p K\left(\frac{p}{\Lambda}\right)\frac{\delta S}{\delta\phi(p)}\mathcal{O}(p)$ is the change of the action under an infinitesimal change of fields:

$$\delta\phi(p) = K\left(\frac{p}{\Lambda}\right)\mathcal{O}(p)$$

(c) $\int_p K\left(\frac{p}{\Lambda}\right)\frac{\delta\mathcal{O}(p)}{\delta\phi(p)}$ is the jacobian of the change of fields.

(d) The vanishing of the composite operator

$$\boxed{\Sigma = 0}$$

implies the invariance of the theory under the transformation $\delta\phi$.

(e) $\Sigma = 0$ if $\Sigma \to 0$ as $\Lambda \to \infty$.

(f) $\Sigma = 0$ gives the Ward identities

$$\sum_{i=1}^{n} \langle\, \phi(p_1)\cdots\mathcal{O}(p_i)\cdots\phi(p_n)\,\rangle_\infty = 0$$

Perturbative application to QED

1. QED is defined by

$$S(\Lambda) = S_{\rm free}(\Lambda) + S_{\rm int}(\Lambda)$$

where

$$\begin{aligned} S_{\rm free}(\Lambda) \quad &\equiv \quad -\frac{1}{2}\int_k \frac{1}{K\left(\frac{k}{\Lambda}\right)} A_\mu(k) A_\nu(-k) \left(k^2 \delta_{\mu\nu} - k_\mu k_\nu \left(1 - \frac{1}{\xi}\right)\right) \\ &\quad - \int_k \frac{1}{K\left(\frac{k}{\Lambda}\right)} \bar{c}(-k) c(k) k^2 - \int_p \frac{1}{K\left(\frac{p}{\Lambda}\right)} \bar{\psi}(-p) \left(\not{p} + im\right) \psi(p) \end{aligned}$$

($c, \bar{c}$ are the free Faddeev-Popov ghosts.)

2. The asymptotic behavior of $S_{\rm int}$ is given by

$$
\begin{aligned}
S_{\rm int} \quad \stackrel{\Lambda\to\infty}{\longrightarrow} \quad & \int \Bigg[\frac{1}{2} A_\mu^2 \left(\Lambda^2 a_2(\ln \Lambda/\mu) + m^2 b_2(\ln \Lambda/\mu) \right) \\
& + c_2(\ln \Lambda/\mu) \frac{1}{2} (\partial_\mu A_\nu)^2 + d_2(\ln \Lambda/\mu) \frac{1}{2} (\partial_\mu A_\mu)^2 \\
& + \bar\psi \left\{ a_f(\ln \Lambda/\mu) i \partial\!\!\!/ + b_f(\ln \Lambda/\mu) i m \right\} \psi \\
& + a_3(\ln \Lambda/\mu) \bar\psi A\!\!\!/ \psi + a_4(\ln \Lambda/\mu) \frac{1}{8} (A_\mu^2)^2 \Bigg]
\end{aligned}
$$

3. Seven parameters: $b_2(0)$, $c_2(0)$, $d_2(0)$, $a_f(0)$, $b_f(0)$, $a_3(0)$, $a_4(0)$
   (a) Normalization: $c_2(0) = a_f(0) = b_f(0) = 0$
   (b) The remaining four are fixed by the BRST invariance.

4. **BRST invariance** (where ϵ is grassmannian)

$$\left\{ \begin{array}{ccl} \delta_\epsilon A_\mu(k) & \equiv & K\left(\frac{k}{\Lambda}\right) k_\mu \epsilon c(k) \\ \delta_\epsilon c(k) & \equiv & 0 \\ \delta_\epsilon \bar{c}(-k) & \equiv & K\left(\frac{k}{\Lambda}\right) \frac{-1}{\xi} k_\mu [A_\mu](-k) \\ \delta_\epsilon \psi(p) & \equiv & K\left(\frac{p}{\Lambda}\right)\ e \int_k \epsilon c(k) [\psi](p-k) \\ \delta_\epsilon \bar{\psi}(-p) & \equiv & K\left(\frac{p}{\Lambda}\right)\ (-e) \int_k \epsilon c(k) [\bar{\psi}](-p-k) \end{array} \right.$$

This conforms with the general form of symmetry transformation.

The BRST invariance is given as

$$\Sigma \equiv \delta_\epsilon S + \underbrace{\Delta_\epsilon S}_{\text{jacobian}} = 0$$

Perturbative application to YM theories

1. A general YM theory is defined by

$$S(\Lambda) = S_{\mathrm{free}}(\Lambda) + S_{\mathrm{int}}(\Lambda)$$

where

$$\begin{aligned} S_{\mathrm{free}}(\Lambda) \equiv & -\frac{1}{2}\int_k \frac{1}{K\left(\frac{k}{\Lambda}\right)} A^a_\mu(k) A^a_\nu(-k) \left(k^2 \delta_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu\right) \\ & - \int_k \frac{1}{K\left(\frac{k}{\Lambda}\right)} \bar{c}^a(-k) c^a(k) k^2 \end{aligned}$$

2. For $SU(2)$ ($f^{abc} = \epsilon^{abc}$), the asymptotic behavior of $S_{\rm int}$ is given by

$$
\begin{aligned}
S_{\rm int}(\Lambda) \stackrel{\Lambda\to\infty}{\longrightarrow} \int \Big[& a_0(\ln\Lambda/\mu)\Lambda^2 \frac{1}{2}(A^a_\mu)^2 \\
& + a_1(\ln\Lambda/\mu)\frac{1}{2}(\partial_\mu A^a_\nu)^2 + a_2(\ln\Lambda/\mu)\frac{1}{2}(\partial_\mu A^a_\mu)^2 \\
& + a_3(\ln\Lambda/\mu) g\epsilon^{abc}\partial_\mu A^a_\nu A^b_\mu A^c_\nu \\
& + a_4(\ln\Lambda/\mu)\frac{g^2}{4} A^a_\mu A^a_\mu A^b_\nu A^b_\nu + a_5(\ln\Lambda/\mu)\frac{g^2}{4} A^a_\mu A^a_\nu A^b_\mu A^b_\nu \\
& + a_6(\ln\Lambda/\mu)\frac{1}{i}\partial_\mu \bar{c}^a \partial_\mu c^a + a_7(\ln\Lambda/\mu) g\epsilon^{abc}\frac{1}{i}\partial_\mu \bar{c}^a A^b_\mu c^c \Big]
\end{aligned}
$$

3. Seven parameters: $a_1(0), \cdots, a_7(0)$ ($a_0(0)$ cannot be controlled.)
   (a) Normalization: $a_1(0) = a_3(0) = a_6(0) = 0$
   (b) The remaining four parameters are fixed by the BRST invariance.

4. **BRST invariance**

$$\left\{ \begin{array}{rcl} \delta_\epsilon A^a_\mu(k) & \equiv & \epsilon K\left(\frac{k}{\Lambda}\right)\left(k_\mu [c^a](k) + \mathcal{A}^a_\mu(k)\right) \\ \delta_\epsilon \bar{c}^a(-k) & \equiv & \epsilon K\left(\frac{k}{\Lambda}\right)\frac{-1}{\xi} k_\mu [A^a_\mu](-k) \\ \delta_\epsilon c^a(k) & \equiv & \epsilon K(k) \mathcal{C}^a(k) \end{array} \right.$$

where $\mathcal{A}^a_\mu$ and $\mathcal{C}^a$ are composite operators, satisfying the following asymptotic conditions:

$$\left\{ \begin{array}{lcl} \mathcal{A}^a_\mu(k) & \stackrel{\Lambda\to\infty}{\longrightarrow} & a_6(\ln\Lambda/\mu) k_\mu c^a(k) + a_7(\ln\Lambda/\mu)\frac{g}{i}\epsilon^{abc}(A^b_\mu c^c)(k) \\ \mathcal{C}^a(k) & \stackrel{\Lambda\to\infty}{\longrightarrow} & a_8(\ln\Lambda/\mu)\frac{g}{2i}\epsilon^{abc}(c^b c^c)(k) \end{array} \right.$$

(a) $a_8(0)$ is the eighth parameter of the theory. ($a_6(0)$ and $a_7(0)$ are part of the parameters of $S_{\rm int}$.)

(b) $\mathcal{A}^a_\mu(k)$ satisfies the ghost equation of motion:

$$k_\mu \mathcal{A}^a_\mu(k) = -\frac{\overrightarrow{\delta}}{\delta \bar{c}^a(-k)} S_{\rm int}$$

(c) The BRST invariance

$$\Sigma \equiv \delta_\epsilon S + \underbrace{\Delta_\epsilon S}_{\rm jacobian} = 0$$

determines $a_8(0)$ as well.

5. **AF formalism** (Batalian-Vilkovisky)

(a) It is non-trivial to show the possibility of satisfying the BRST invariance by tuning the five parameters (including $a_8(0)$) of the theory.

(b) For each field, introduce an external field, called an **antifield**, that generates the BRST transformation.

field	dim	ghost	stat	antifield	dim	ghost	stat
A_μ^a	1	0	b	A_μ^{a*}	2	-1	f
c^a	1	1	f	c^{a*}	2	-2	b
$\bar{c}^a$	1	-1	f	$\bar{c}^{a*}$	2	0	b

(c) The AF dependence of the full action $\mathcal{S}$ is introduced so that the BRST transformation is given by

$$\left\{ \begin{array}{rcl} \delta A_\mu^a(k) &=& K\left(\frac{k}{\Lambda}\right) \frac{\overrightarrow{\delta}}{\delta A_\mu^{a*}(-k)} \mathcal{S} \\ \delta c^a(k) &=& K\left(\frac{k}{\Lambda}\right) \frac{\delta \mathcal{S}}{\delta c^{a*}(-k)} \\ \delta \bar{c}^a(-k) &=& K\left(\frac{k}{\Lambda}\right) \frac{\delta \mathcal{S}}{\delta \bar{c}^{a*}(k)} \end{array} \right.$$

(d) The BRST invariance is now written as

$$\begin{aligned}\Sigma \quad \equiv \quad & \int_k K\left(\frac{k}{\Lambda}\right)\left[\frac{\overrightarrow{\delta}}{\delta A_\mu^{a*}(-k)}\mathcal{S}\cdot\frac{\delta\mathcal{S}}{\delta A_\mu^a(k)} + \frac{\overrightarrow{\delta}}{\delta A_\mu^{a*}(-k)}\frac{\delta\mathcal{S}}{\delta A_\mu^a(k)}\right. \\ & + \frac{\delta\mathcal{S}}{\delta\bar{c}^{a*}(k)}\frac{\overrightarrow{\delta}}{\delta\bar{c}^a(-k)}\mathcal{S} + \frac{\delta}{\delta\bar{c}^{a*}(k)}\frac{\overrightarrow{\delta}}{\delta\bar{c}^a(-k)}\mathcal{S} \\ & \left. - \frac{\delta\mathcal{S}}{\delta c^{a*}(-k)}\cdot\mathcal{S}\frac{\overleftarrow{\delta}}{\delta c^a(k)} - \frac{\delta\mathcal{S}}{\delta c^{a*}(-k)}\frac{\overleftarrow{\delta}}{\delta c^a(k)}\right] = 0\end{aligned}$$

Σ is a composite operator. Hence, $\Sigma = 0$ if $\Sigma \to 0$ as $\Lambda \to \infty$.

(e) $\mathcal{S}$ satisfies the same ERG differential equation as S. Its asymptotic behavior is determined by the same seven parameters $a_1(0), \cdots, a_7(0)$ that define S, in addition to $a_8(0)$.

(f) The BRST transformation δ_Q is defined by

$$\delta_Q \mathcal{O} \equiv \int_q K\left(\frac{q}{\Lambda}\right) \Bigg[$$

$$\frac{\delta \mathcal{S}}{\delta A_\mu^a(-q)} \cdot \frac{\overrightarrow{\delta}}{\delta A_\mu^{a*}(q)} \mathcal{O} + \frac{\overrightarrow{\delta}}{\delta A_\mu^{a*}(q)} \mathcal{S} \cdot \frac{\delta}{\delta A_\mu^a(-q)} \mathcal{O} + \frac{\overrightarrow{\delta}}{\delta A_\mu^{a*}(q)} \frac{\delta \mathcal{O}}{\delta A_\mu^a(-q)}$$

$$+ \frac{\overrightarrow{\delta}}{\delta \bar{c}^a(-q)} \mathcal{S} \cdot \frac{\delta}{\delta \bar{c}^{a*}(q)} \mathcal{O} + \frac{\delta \mathcal{S}}{\delta \bar{c}^{a*}(q)} \frac{\overrightarrow{\delta}}{\delta \bar{c}^a(-q)} \mathcal{O} + \frac{\delta}{\delta \bar{c}^{a*}(k)} \frac{\overrightarrow{\delta}}{\delta \bar{c}^a(-k)} \mathcal{O}$$

$$\frac{\overrightarrow{\delta}}{\delta c^a(-q)} \mathcal{S} \cdot \frac{\delta}{\delta c^{a*}(q)} \mathcal{O} + \frac{\delta \mathcal{S}}{\delta c^{a*}(q)} \frac{\overrightarrow{\delta}}{\delta c^a(-q)} \mathcal{O} + \frac{\delta}{\delta c^{a*}(k)} \frac{\overrightarrow{\delta}}{\delta c^a(-k)} \mathcal{O} \Bigg]$$

(g) If $\Sigma = 0$, then $\delta_Q^2 = 0$, and $\delta_Q \mathcal{O}$ is a composite operator if $\mathcal{O}$ is.

(h) For an arbitrary choice of the eight parameters, Σ still satisfies the algebraic constraint

$$\boxed{\delta_Q \Sigma = 0}$$

Using this, we can show the possibility of tuning the parameters to satisfy the BRST invariance $\Sigma = 0$. (Becchi '93)

(i) AF for QED has been discussed by Igarashi, Itoh, H.S. '07 and Higashi, Itou, Kugo '07

Perturbative application to the WZ model

1. We wish to construct supersymmetric theories without using superfields or auxiliary fields. $\longrightarrow$ No more superspace or dimensional reduction

2. To define the Wess-Zumino model, we introduce

$$S = S_{\rm free} + S_{\rm int}$$

Using the two-component spinor notation ($\bar{\chi} \equiv \chi^T \sigma_y$)

$$\begin{aligned} S_{\rm free} \quad \equiv \quad & - \int_p \frac{1}{K(p/\Lambda)} \Big[\bar{\phi}(-p)\phi(p)(p^2 + |m|^2) \\ & + \bar{\chi}_L(-p)\sigma_\mu i p_\mu \chi_R(p) + \frac{m}{2}\bar{\chi}_R(-p)\chi_R(p) + \frac{\bar{m}}{2}\bar{\chi}_L(-p)\chi_L(p) \Big] \end{aligned}$$

3. The ERG differential equation for $S_{\rm int}$:

$$
\begin{aligned}
-\Lambda \frac{\partial}{\partial \Lambda} S_{\rm int} = \int_p \frac{\Delta(p/\Lambda)}{p^2+|m|^2} \Bigg[& \frac{\delta S_{\rm int}}{\delta \phi(p)} \frac{\delta S_{\rm int}}{\delta \bar{\phi}(-p)} + \frac{\delta^2 S_{\rm int}}{\delta \phi(p) \delta \bar{\phi}(-p)} \\
& - {\rm Tr}\, (-ip\cdot\bar{\sigma}) \left\{ \frac{\overrightarrow{\delta}}{\delta \bar{\chi}_L(-p)} S_{\rm int} \cdot S_{\rm int} \frac{\overleftarrow{\delta}}{\delta \chi_R(p)} + \frac{\overrightarrow{\delta}}{\delta \bar{\chi}_L(-p)} S_{\rm int} \frac{\overleftarrow{\delta}}{\delta \chi_R(p)} \right\} \\
& - \bar{m} {\rm Tr}\, \left\{ \frac{\overrightarrow{\delta}}{\delta \bar{\chi}_R(-p)} S_{\rm int} \cdot S_{\rm int} \frac{\overleftarrow{\delta}}{\delta \chi_R(p)} + \frac{\overrightarrow{\delta}}{\delta \bar{\chi}_R(-p)} S_{\rm int} \frac{\overleftarrow{\delta}}{\delta \chi_R(p)} \right\} \\
& - m {\rm Tr}\, \left\{ \frac{\overrightarrow{\delta}}{\delta \bar{\chi}_L(-p)} S_{\rm int} \cdot S_{\rm int} \frac{\overleftarrow{\delta}}{\delta \chi_L(p)} + \frac{\overrightarrow{\delta}}{\delta \bar{\chi}_L(-p)} S_{\rm int} \frac{\overleftarrow{\delta}}{\delta \chi_L(p)} \right\} \Bigg]
\end{aligned}
$$

4. $S_{\rm int}$ is the solution with the following asymptotic behavior:

$$S_{\rm int}(\Lambda) \quad \stackrel{\Lambda\to\infty}{\longrightarrow} \quad \int \Bigg[z_1 \bar{\chi}_L \sigma_\mu \partial_\mu \chi_R + z_2 \left(\frac{m}{2} \bar{\chi}_R \chi_R + \frac{\bar{m}}{2} \bar{\chi}_L \chi_L \right)$$

$$+ z_3 \partial_\mu \bar{\phi} \partial_\mu \phi + (a_4 \Lambda^2 + z_4 |m|^2) |\phi|^2$$

$$+ (-1 + z_5) \left(g \phi \frac{1}{2} \bar{\chi}_R \chi_R + \bar{g} \bar{\phi} \frac{1}{2} \bar{\chi}_L \chi_L \right)$$

$$+ (-1 + z_6) \left(m \phi \frac{\bar{g}}{2} \bar{\phi}^2 + \bar{m} \bar{\phi} \frac{g}{2} \phi^2 \right) + (-1 + z_7) \frac{|g|^2}{4} |\phi|^4$$

$$+ z_8 \left(g^2 \bar{m}^2 \phi^2 + \bar{g}^2 m^2 \bar{\phi}^2 \right)$$

$$+ (\Lambda^2 a_9 + |m|^2 z_9) \left(g \bar{m} \phi + \bar{g} m \bar{\phi} \right) \Bigg]$$

where $a_{4,9}$ and $z_i \, (i = 1, \cdots, 9)$ are all functions of $|g|^2$ and $\ln \Lambda/\mu$.

5. The supersymmetry transformation is defined by

$$\left\{ \begin{array}{rcl} \delta\phi(p) & \equiv & \bar{\xi}_R[\chi_R](p) + \eta_\mu i p_\mu [\phi](p) \\ \delta\bar{\phi}(p) & \equiv & \bar{\xi}_L[\chi_L](p) + \eta_\mu i p_\mu [\bar{\phi}](p) \\ \delta\chi_R(p) & \equiv & \bar{\sigma}_\mu \xi_L i p_\mu [\phi](p) - \left(\bar{m}[\bar{\phi}](p) + \bar{g} \cdot \left[\frac{\bar{\phi}^2}{2}\right](p) \right) \xi_R + \eta_\mu i p_\mu [\chi_R](p) \\ \delta\chi_L(p) & \equiv & \sigma_\mu \xi_R i p_\mu [\bar{\phi}](p) - \left(m[\phi](p) + g \cdot \left[\frac{\phi^2}{2}\right](p) \right) \xi_L + \eta_\mu i p_\mu [\chi_L](p) \end{array} \right.$$

where $\xi_{R,L}$ are **grassmann constant spinors**. An **infinitesimal constant vector** η_μ, generating a translation, is added for closing the supersymmetry algebra.

6. The composite operators $[\phi^2/2], [\bar{\phi}^2/2]$ are defined by

$$\left\{ \begin{array}{l} \left[\frac{\phi^2}{2}\right](p) \stackrel{\Lambda\to\infty}{\longrightarrow} (1+z_{10})\frac{\phi^2}{2}(p) + z_{11}\bar{g}m\phi(p) + z_{12}\bar{g}^2 m^2 \cdot (2\pi)^4 \delta^{(4)}(p) \\ \left[\frac{\bar{\phi}^2}{2}\right](p) \stackrel{\Lambda\to\infty}{\longrightarrow} (1+z_{10})\frac{\bar{\phi}^2}{2}(p) + z_{11} g\bar{m}\bar{\phi}(p) + z_{12} g^2 \bar{m}^2 \cdot (2\pi)^4 \delta^{(4)}(p) \end{array} \right.$$

where $z_{10,11,12}$ are functions of $|g|^2$ and $\ln \Lambda/\mu$.

7. Altogether, the theory has **twelve** parameters.

8. The parameters are constrained by the invariance:

$$\Sigma \equiv \int_p K(p/\Lambda) \Bigg[\delta\phi(p) \frac{\delta S}{\delta\phi(p)} + \delta\bar{\phi}(p) \frac{\delta S}{\delta\bar{\phi}(p)} + \frac{\delta}{\delta\phi(p)} \delta\phi(p) + \frac{\delta}{\delta\bar{\phi}(p)} \delta\bar{\phi}(p)$$
$$+ S \frac{\overleftarrow{\delta}}{\delta\chi_R(p)} \delta\chi_R(p) + S \frac{\overleftarrow{\delta}}{\delta\chi_L(p)} \delta\chi_L(p) - \mathrm{Tr}\ \delta\chi_R(p) \frac{\overleftarrow{\delta}}{\delta\chi_R(p)} - \mathrm{Tr}\ \delta\chi_L(p) \frac{\overleftarrow{\delta}}{\delta\chi_L(p)} \Bigg] = 0$$

(a) z_1, z_3, z_5 — common normalization of scalars and spinors, and normalization of $|m|$ and g

(b) The remaining nine are fixed by $\Sigma = 0$.

9. The AF formalism is being worked out. (K. Ülker, H.S.)

Concluding remarks

1. More perturbative applications wanted (only details to be worked out)

2. How can we formulate ERG on a lattice?