一般化された不確定性関係に基づいた場の量子論

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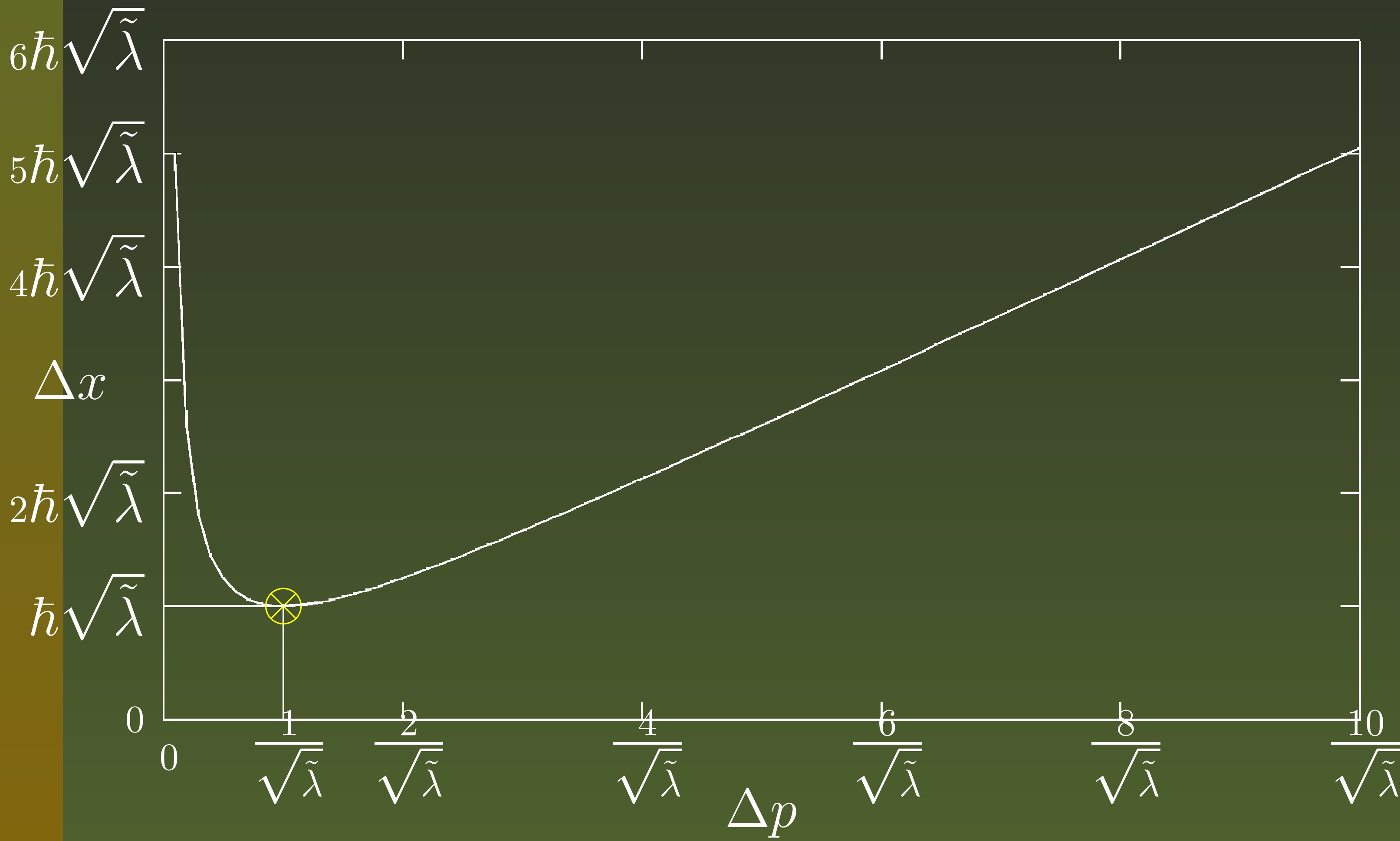
理研

Generalized uncertainty relation



Generalized uncertainty relation

$$\Delta x \Delta p \geq \frac{1}{2}(1 + \tilde{\lambda}(\Delta p)^2 + \tilde{\lambda} < \hat{p} >^2)$$

- High energy string scattering — Gross,Mende Phys.Lett.B197(1987)129, K.Konishi et.al Phys.Lett.B234(1990)276

$$\hbar\sqrt{\tilde{\lambda}} \sim \text{String scale}$$

- Gedanken exp. of Black hole — Maggiore Phys.Lett.B304(1993)65
- Snyder's quantized space-time (de Sitter) — Snyder Phys.Rev.71(1947)38
- Symmetry of massless particles — Filho hep-th/0505183

Deformed canonical commutation relation

$$[\hat{x}, \hat{p}] = i\hbar(1 + \tilde{\lambda}\hat{p}^2)$$

Outline

- Introduction
- Hilbert space in the first quantization
- $1+1$-dimensional quantum field theory – free scalar field
  - Canonical formalism
  - Difficulty in higher dimensional space-time
- Classical mechanics
- $1+d$-dimensional quantum field theory
- High temperature property
- Conclusion and discussions

One particle state

$$\left[\hat{x},\hat{k}\right] = i(1+\lambda\hat{k}^2), \quad \text{here, } \hat{k} \equiv \frac{\hat{p}}{\hbar}, \lambda \equiv \hbar^2\tilde{\lambda},$$

$$\begin{aligned}\left(\hat{x}- < \hat{x} > + \frac{< [\hat{x},\hat{k}] >}{2(\Delta k)^2}(\hat{k}- < \hat{k} >)\right)|\Psi > &= 0 \\ \left(i(1+\lambda k^2)\frac{\partial}{\partial k} - < \hat{x} > + i\frac{1+\lambda(\Delta k)^2 + \lambda < \hat{k} >^2}{2(k- < \hat{k} >)}\right)\Psi(k) &= 0\end{aligned}$$

'Maximal localization state' $\Delta k = \frac{1}{\sqrt{\lambda}}, \Delta x = \sqrt{\lambda}, < \hat{k} >= 0, < x >\equiv \xi$

(A.Kempf et. al. PRD52(95)1108)

$$\begin{aligned}\Psi_\xi(k) &= N(1+\lambda k^2)^{-\frac{1}{2}}\exp(-i\frac{\xi}{\sqrt{\lambda}}\tan^{-1}(\sqrt{\lambda}k)), \\ &\equiv < k|\xi >\end{aligned}$$

Hilbert space

- Completeness of $|k\rangle$

$$\mathbb{1} \;=\; \int \frac{dk}{1+\lambda k^2}|k\rangle\langle k|$$

- Orthogonality of $|k\rangle$

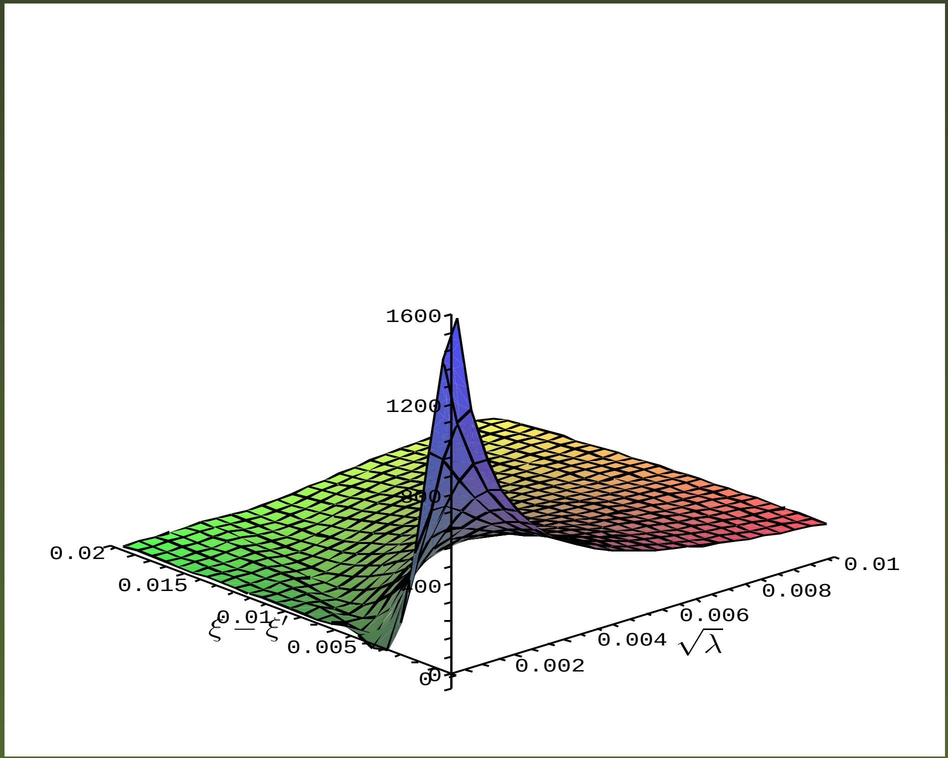
$$\langle k|k'\rangle \;=\; (1+\lambda k^2)\delta(k-k').$$

- Completeness of $|\xi\rangle$

$$\mathbb{1} = \int \frac{d\xi}{2\pi N^2}(1+\lambda\hat{k}^2)|\xi\rangle\langle\xi|$$

Orthogonality of $|\xi\rangle$

$$\langle\xi|\xi'\rangle = \int \frac{dk}{1+\lambda k^2}\Psi^*_\xi(k)\Psi_{\xi'}(k) = \frac{N^2}{2\sqrt{\lambda}}\frac{\sin(\frac{\pi(\xi'-\xi)}{2\sqrt{\lambda}})}{(\frac{\xi'-\xi}{2\sqrt{\lambda}})-(\frac{\xi'-\xi}{2\sqrt{\lambda}})^3}$$



ξ表示とk表示の『Fourier 変換』

$$
\begin{aligned}
\phi(\xi) &\equiv\ <\xi|\phi>, \phi(k) \equiv <k|\phi> \\
\phi(\xi) &= \int \frac{dk}{1+\lambda k^2} \Psi^*_\xi(k)\phi(k) \\
<\xi|\phi> &= \int \frac{dk}{1+\lambda k^2} <\xi|k><k|\phi>, \\
\phi(k) &= \int \frac{d\xi}{2\pi N^2}(1+\lambda k^2)\Psi_\xi(k)\phi(\xi) \\
<k|\phi> &= \int \frac{d\xi}{2\pi N^2}(1+\lambda k^2) <k|\xi><\xi|\phi>
\end{aligned}
$$

この Fourier 変換は、任意の $\phi(\xi)$ の上では成立しない。

$\phi(\xi) = \sum a(k)\Psi^*_\xi(k)$ すなわち、今考えている Hilbert 空間において well-defined となっている。

$1+1$ dimensional quantum field theory –free scalar (T.Matsuo and S.Y.Mod. Phys. Lett.21(2006)1285)

- Klein-Gordon equation and fields

$$
\begin{aligned}
0 &= ((\partial_t)^2 + \frac{F(k)+m^2}{\hbar^2})\hat{\Phi}(k,t), \\
\hat{\Phi}(k,t) &\equiv \sqrt{\frac{\hbar}{2\omega(k)}}\hat{\phi}(k)\exp(-i\omega(k)t) + \sqrt{\frac{\hbar}{2\omega(-k)}}\hat{\phi}^\dagger(-k)\exp(i\omega(k)t), \\
\hat{\Phi}(\xi,t) &\equiv \int \frac{dk}{1+\lambda k^2}\sqrt{\frac{\hbar}{2\omega(k)}}\left\{\Psi^*_\xi(k)\hat{\phi}(k)e^{-i\omega(k)t} + \Psi_\xi(p)\hat{\phi}^\dagger(k)e^{i\omega(k)t}\right\}. \\
\hat{\Pi}(\xi,t) &\equiv \partial_t\hat{\Phi}(\xi,t), \\
\omega(k) &\equiv \frac{\sqrt{F(k)+m^2}}{\hbar} \quad F(k) \equiv k^2.
\end{aligned}
$$

- Creation-annihilation operator

$$
\begin{aligned}
\left[\hat{\phi}(k),\hat{\phi}^\dagger(k')\right] &= (1+\lambda k^2)\delta(k-k'), \\
&\leftarrow \langle k|k'\rangle = (1+\lambda k^2)\delta(k-k').
\end{aligned}
$$

$1+1$ dimensional quantum field theory –free scalar

- Canonical commutation relation (Non-locality)

$$\left[\hat{\Phi}(k,t),\hat{\Pi}(k',t)\right] = i\hbar(1+\lambda k^2)\delta(k+k')$$
$$\left[\hat{\Phi}(\xi,t),\hat{\Pi}(\xi',t)\right] = i\hbar\langle\xi|\xi'\rangle$$

- Hamiltonian and Heisenberg equation

$$H \equiv \int \frac{dk}{1+\lambda k^2}\hbar\omega(k)(\hat{\phi}^\dagger(k)\hat{\phi}(k)+\text{const.}),$$
$$\left[\hat{\Phi}(\xi,t),H\right] = i\hbar\hat{\Pi}(\xi,t) = i\hbar\partial_t\hat{\Phi}(\xi,t)$$

$1+1$ dimensional quantum field theory –free scalar

- 高次元時空での問題点

$$
\begin{aligned}
\left[\hat{x}^i, \hat{k}_j\right] &= i\hbar(1+\tilde{\lambda}\hat{\boldsymbol{p}}^2)\delta^i_j, \\
\left[\hat{x}^i, \hat{x}^j\right] &= -2i\hbar\tilde{\lambda}(1+\tilde{\lambda}\hat{\boldsymbol{p}}^2)\hat{L}^{ij}. \\
\hat{L}^{ij} &\equiv \frac{1}{1+\tilde{\lambda}\hat{\boldsymbol{p}}^2}(\hat{x}^i\hat{p}^j - \hat{x}^j\hat{p}^i)
\end{aligned}
$$

$$
\left(i\hbar(1+\tilde{\lambda}\boldsymbol{p}^2)\frac{\partial}{\partial p_i} - \langle \hat{x}^i \rangle + i\hbar \frac{1+\tilde{\lambda}(\Delta p_i)^2 + \tilde{\lambda}\langle p_i \rangle^2}{2\Delta p} \right) \psi(p) = 0.
$$

同時対角化可能は s 波のみ。パラメーター ξ が定義できない。

Classical mechanics

(T.Takeuchi et. al PRD65(2002)125028)

- Poisson 括弧による解析力学

$$\dot{x}^i = \{x^i, H\} = (1+\tilde{\lambda}\mathbf{p}^2)\frac{\partial H}{\partial p^i} + 2\tilde{\lambda}p^i x^j \frac{\partial H}{\partial x^j} - 2\tilde{\lambda}p^j x^i \frac{\partial H}{\partial x^j},$$

$$\dot{p}^i = \{p^i, H\} = -(1+\tilde{\lambda}\mathbf{p}^2)\frac{\partial H}{\partial x^i}.$$

- 位相空間の不変体積

Liouville の定理から 不変体積が以下のようになる

$$(1+\tilde{\lambda}\mathbf{p}^2)^{-d}\prod_i \Delta x^i \Delta p^i$$

$$\text{量子化} \quad \rightarrow \quad \frac{\prod_i \Delta x^i \Delta p^i}{(1+\tilde{\lambda}\mathbf{p}^2)^d} = (2\pi\hbar)^d.$$

$1+d$-dimensional quantum field theory –free scalar

- Completeness of $|k\rangle$

$$\mathbb{1} \;=\; \int \frac{d^dk}{(1+\lambda\boldsymbol{k}^2)^d}|k\rangle\langle k|$$

- Orthogonality of $|k\rangle$

$$\langle k|k'\rangle \;=\; (1+\lambda\boldsymbol{k}^2)^d\delta^d(k-k').$$

測度 $\frac{d^dk}{(1+\lambda\boldsymbol{k}^2)^d}$ での2乗可積分函数は、

$\langle k|\rho\rangle = \sqrt{\frac{(1+\lambda\boldsymbol{k}^2)^d}{(2\pi)^d}}\exp(-ik\rho)$ で展開可能。

$1+d$-dimensional quantum field theory –free scalar

$|\rho\rangle$ という状態は、位置演算子

$$\hat{x}^i = (1+\tilde{\lambda}\mathbf{k}^2)^{\frac{1+d}{2}}\partial_{k_i}(1+\tilde{\lambda}\mathbf{k}^2)^{\frac{1-d}{2}}$$

ではなく別のエルミート演算子 (同時対角化可能)

$$\hat{\rho}^i = (1+\tilde{\lambda}\mathbf{k}^2)^{\frac{d}{2}}\partial_{k_i}(1+\tilde{\lambda}\mathbf{k}^2)^{-\frac{d}{2}},$$
$$0 = \left[\hat{\rho}^i, \hat{\rho}^j\right].$$

の固有状態という意味で完全系を張っている。

- Completeness and orthogonality of $|\rho\rangle$

$$\mathbb{1} = \int d^d\rho|\rho\rangle\langle\rho|,$$
$$\langle\rho|\rho'\rangle = \delta^d(\rho-\rho').$$

$1+d$ dimensional quantum field theory

- Fields (ρ はパラメーターとして使える。)

$$
\begin{aligned}
\hat{\Phi}(k,t) &\equiv \sqrt{\frac{\hbar}{2\omega(k)}}\hat{\phi}(k)\exp(-i\omega(k)t) + \sqrt{\frac{\hbar}{2\omega(-k)}}\hat{\phi}^\dagger(-k)\exp(i\omega(k)t), \\
\hat{\Phi}(\rho,t) &\equiv \int \frac{dk}{((2\pi)(1+\lambda\boldsymbol{k}^2))^{\frac{d}{2}}}\sqrt{\frac{\hbar}{2\omega(k)}}\left\{e^{-i\omega t+ik\rho}\hat{\phi}(k)+e^{i\omega t-ik\rho}\hat{\phi}^\dagger(k)\right\}. \\
\hat{\Pi}(\xi,t) &\equiv \partial_t\hat{\Phi}(\xi,t) \\
\omega(k) &\equiv \frac{\sqrt{F(k)+m^2}}{\hbar} \quad F(k)=\mathbf{k}^2
\end{aligned}
$$

- Creation-annihilation operator

$$
\begin{aligned}
\left[\hat{\phi}(k),\hat{\phi}^\dagger(k')\right] &= (1+\lambda\boldsymbol{k}^2)^d\delta^d(k-k'), \\
&\leftarrow \langle k|k'\rangle = (1+\lambda\boldsymbol{k}^2)^d\delta^d(k-k').
\end{aligned}
$$

Canonical commutation relations and Hamiltonian

- Canonical commutation relation (not non-locality)

$$\left[\hat{\Phi}(k,t), \hat{\Pi}(k',t)\right] = i\hbar(1+\lambda\boldsymbol{k}^2)^d \delta^d(k+k')$$
$$\left[\hat{\Phi}(\rho,t), \hat{\Pi}(\rho',t)\right] = i\hbar\delta^d(\rho-\rho')$$

- Hamiltonian

$$H \equiv \int \frac{d^d k}{(1+\lambda\boldsymbol{k}^2)^d} \hbar\omega(k)\hat{\phi}^\dagger(k)\hat{\phi}(k)$$
$$= \int \frac{d^d k}{(1+\lambda\boldsymbol{k}^2)^d} \frac{1}{2}\left\{\dot{\hat{\Phi}}(-k,t)\dot{\hat{\Phi}}(k,t) + \omega^2\hat{\Phi}(-k,t)\hat{\Phi}(k,t)\right\} - \sum_{\text{states}} \frac{\hbar\omega}{2}$$

Path integral formalism in $1+d$-dimensions

多粒子の Hilbert 空間の完全系を挟むことにより経路積分表示が得られる。

$$Z = \int \mathcal{D}\Phi\mathcal{D}\Pi \exp(\frac{i}{\hbar}\int dt \frac{d^dk}{(1+\lambda\boldsymbol{k}^2)^d}\left\{\dot{\Phi}(k,t)\Pi(k,t) - \frac{1}{2}(\Pi\Pi - \omega^2\Phi\Phi)\right\})$$

$$= \int \mathcal{D}\Phi \exp(\frac{i}{\hbar}\int dt \frac{d^dk}{(1+\lambda\boldsymbol{k}^2)^d}\frac{1}{2}\left\{\dot{\Phi}(-k,t)\dot{\Phi}(k,t) - \omega^2\Phi(-k,t)\Phi(k,t)\right\})$$

Path integral measure is

$$\mathcal{D}\Phi\mathcal{D}\Pi \equiv \prod_{j=1}^{N-1}\prod_{k}\frac{\Delta^d k}{(1+\lambda\boldsymbol{k}^2)^d}d\Phi(k,t_j)d\Pi(k,t_j).$$

Finite temperature field theory

実スカラー場の有限温度での自由エネルギー密度は

$$
\begin{aligned}
F(\beta) &\equiv -\frac{1}{\beta V}\ln Z(\beta) \\
&= \int \frac{d^d k}{(1+\lambda \boldsymbol{k}^2)^d}\left(\frac{\hbar\omega}{2} + \frac{1}{\beta}\ln(1-e^{-\hbar\omega\beta})\right) \\
&= \frac{1}{\beta}\int \frac{d^d k}{(1+\lambda \boldsymbol{k}^2)^d}\ln(2\sinh(\frac{\hbar\beta\omega}{2})).
\end{aligned}
$$

high-temperature expansion

- massless particle

$$F = -\zeta(1)\frac{d\pi^{\frac{d+1}{2}}\Gamma(\frac{d}{2})}{(4\pi)^d}\Gamma(\frac{d+1}{2})\Gamma(\frac{d+2}{2})\lambda^{-\frac{d}{2}}\beta^{-1} + \mathcal{O}(\beta^0).$$

- massive particle ($m \sim \frac{1}{\sqrt{\lambda}}$)

$$F = -\zeta(1)\frac{d\pi^{\frac{d+1}{2}}\Gamma(\frac{d}{2})}{(4\pi)^d}\Gamma(\frac{d+1}{2})\Gamma(\frac{d+2}{2})\lambda^{-\frac{d}{2}}\beta^{-1} + \mathcal{O}(\beta^0).$$

$F \propto T^1$

Remark:

in ordinary QFT $F \propto T^{(d+1)}$

弦理論から予想される自由エネルギーの温度依存性

- T-duality

$$
\begin{aligned}
\frac{1}{T}F(T) &= \frac{T}{(T_d)^2}F(\frac{T_d^2}{T}), T_d \equiv \frac{1}{2\pi\sqrt{\alpha'}}, \\
F(T) &\sim a(T_d)T^2 + b(T_d)T + a(T_d)T_d^2. \quad T \sim T_d\text{付近で}
\end{aligned}
$$

- 状態密度の Hardy-Ramanujan 的振舞い

$$
\begin{aligned}
W(E) &= \exp(aE)E^b, \\
E &= \frac{bT}{1-aT}, \\
F(T) &= bT - bT\log(\frac{bT}{1-aT}) \sim T\log T \,\text{or}\, T^1.
\end{aligned}
$$

自由エネルギーの振舞い (数値計算)

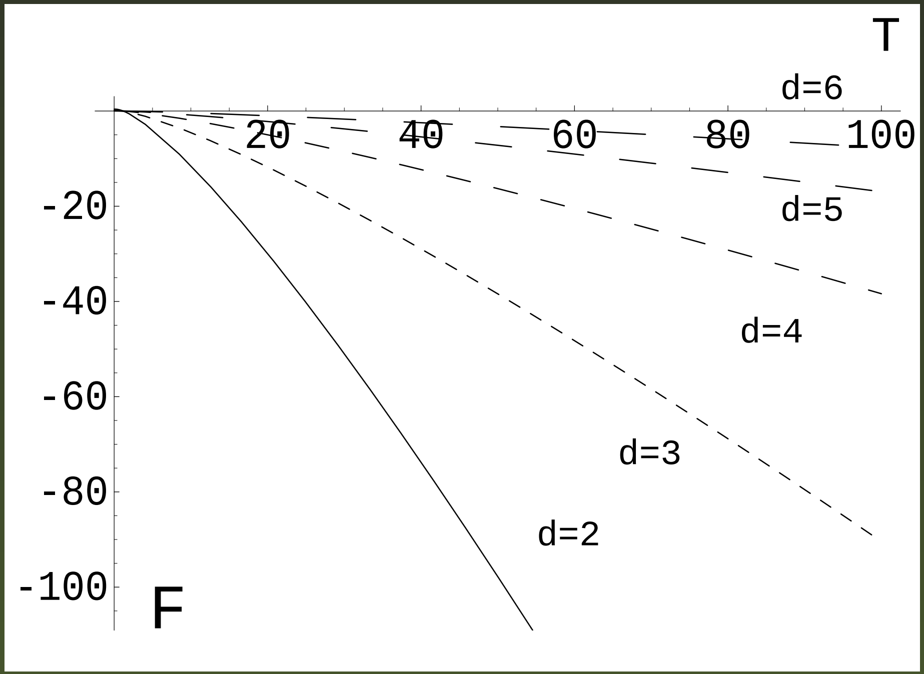


Figure 1: massless boson (d=2,3,4,5,6) 注 rescale しています。

自由エネルギーの振舞い (数値計算)

前述の結果を $T = 0.8 \times 10^{\alpha} \sim 1.2 \times 10^{\alpha}$ の 21 点でフィッティング

α	-2	-1	0	1	2
1	6.22×10^{-4}	6.23×10^{-4}	7.68×10^{-4}	2.38×10^{-4}	2.49×10^{-5}
T	8.07×10^{-7}	-5.02×10^{-4}	-1.23×10^{-3}	-1.79×10^{-4}	-6.55×10^{-6}
T^2	0.0	1.66×10^{-3}	5.77×10^{-4}	1.19×10^{-5}	1.28×10^{-8}
T^3	0.0	-4.78×10^{-3}	-3.08×10^{-5}	-2.98×10^{-7}	-3.25×10^{-11}
T^4	-8.47×10^{-3}	2.17×10^{-3}	-8.68×10^{-6}	7.99×10^{-9}	8.72×10^{-14}
T^5	2.40×10^{-1}	1.32×10^{-2}	3.55×10^{-6}	-1.50×10^{-10}	-1.66×10^{-16}
T^6	-1.08	-2.24×10^{-2}	-4.37×10^{-7}	1.38×10^{-12}	1.52×10^{-19}
$T \ln T$	3.52×10^{-7}	-2.32×10^{-4}	-9.45×10^{-4}	-9.47×10^{-4}	-1.05×10^{-3}
$T(\ln T)^2$	7.67×10^{-9}	-3.14×10^{-5}	-4.51×10^{-4}	-3.56×10^{-5}	-3.83×10^{-7}

Table 1: $d = 4$

自由エネルギーの振舞い (数値計算)

α	-2	-1	0	1	2
1	4.64×10^{-6}	4.66×10^{-6}	6.14×10^{-6}	1.48×10^{-6}	1.50×10^{-7}
T	2.52×10^{-9}	-8.68×10^{-6}	-1.27×10^{-5}	-1.12×10^{-6}	-3.91×10^{-8}
T^2	-1.37×10^{-7}	2.60×10^{-5}	7.96×10^{-6}	7.54×10^{-8}	7.62×10^{-11}
T^3	3.23×10^{-6}	-6.27×10^{-5}	-9.79×10^{-7}	-1.90×10^{-9}	-1.94×10^{-13}
T^4	-1.49×10^{-4}	8.27×10^{-5}	9.40×10^{-8}	5.12×10^{-11}	5.20×10^{-16}
T^5	1.88×10^{-3}	-7.99×10^{-5}	3.31×10^{-9}	-9.65×10^{-13}	-9.93×10^{-19}
T^6	-2.95×10^{-2}	8.87×10^{-5}	-1.66×10^{-9}	8.86×10^{-15}	8.97×10^{-22}
$T \ln T$	4.13×10^{-9}	-4.14×10^{-6}	-9.54×10^{-6}	-7.72×10^{-6}	-8.38×10^{-6}
$T(\ln T)^2$	2.10×10^{-10}	-5.88×10^{-7}	-4.57×10^{-6}	-2.24×10^{-7}	-2.28×10^{-9}

Table 2: $d = 6$

次元に依らず $\alpha = -1 \sim 0$ で $T^n (n > 2)$ の係数が落ち、段々 $T \ln T$ になる。

Conclusion and discussion

- Field theory in GUP
  - $1+1$-dimensional free scalar
  - $1+d$-dimensional free scalar
- High temperature behaviour
- Other topics
  - Fermion (SUSY)
  - Lorentz invariance
    Other canonical commutation relation
    κ-deformation
  - Interaction
  - Renormalization

Canonical commutation relation in path integral

T^*-product is

$$\langle T^*\Phi(k,l)\Phi(k',l')\rangle \equiv \int \frac{dtdt'}{2\pi} e^{ilt+il't'} \langle T^*\Phi(k,l)\Phi(k',l')\rangle$$
$$= \frac{\hbar}{i}\frac{\delta(l+l')\delta^d(k+k')(1+\lambda \boldsymbol{k}^2)^d}{-l^2+\omega^2-i\varepsilon}.$$

From BJL prescription, we can obtain

$$\left[\hat{\Phi}(k,t), \dot{\hat{\Phi}}(k',t)\right] = i\hbar(1+\lambda k^2)^d \delta^d(k+k').$$

This result is consistent with Hamiltonian formalism.

Generalized uncertainty relation

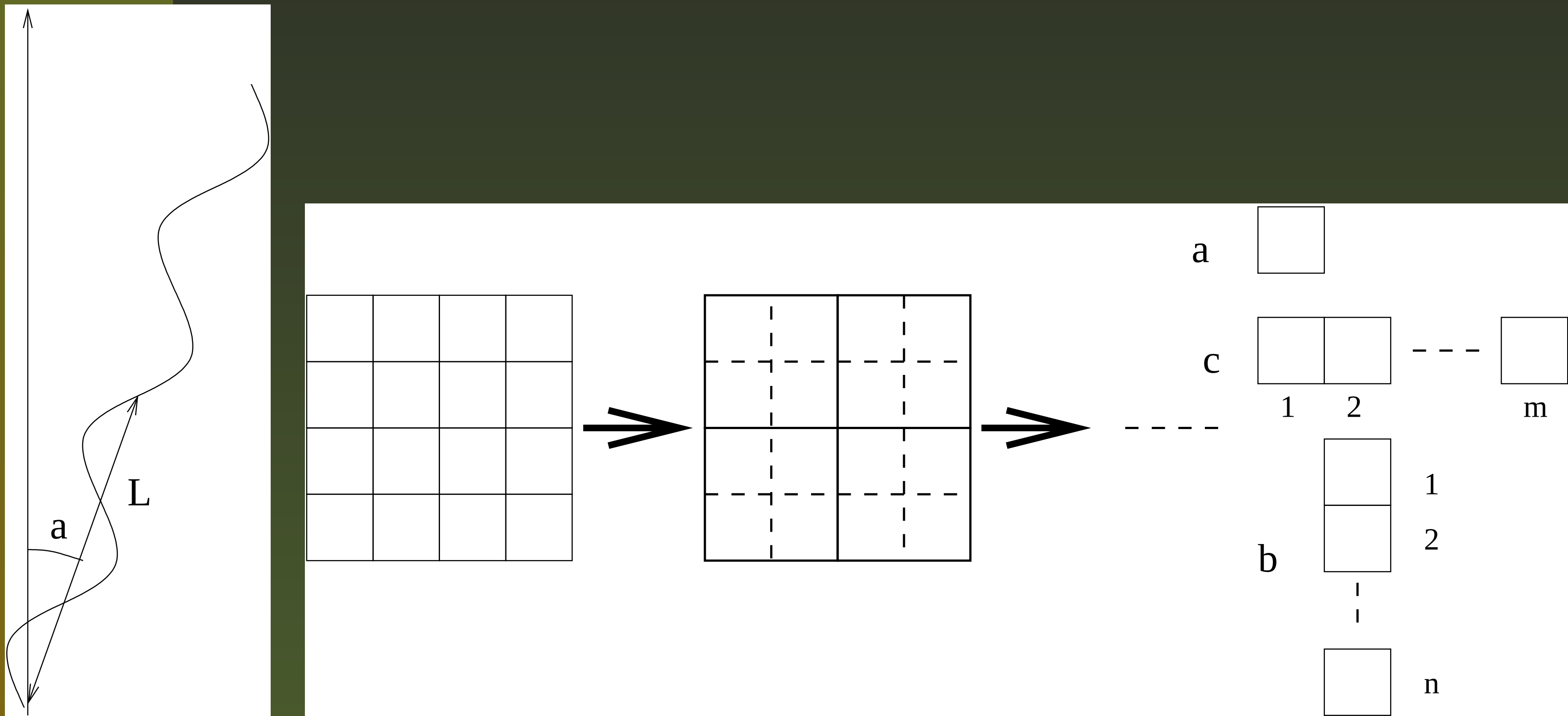


Figure 2: Left:Black Hole Gedanken exp., Right: Migdal-Kadanoff trans. in String

Generalized uncertainty relation

Snyder's quantized space-time (de Sitter)

$$
\begin{aligned}
-\eta^2 &= \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 . \\
x &= ia(\eta_4 \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_4}), \\
p_i &= \frac{\hbar}{a} \frac{\eta_i}{\eta_4},
\end{aligned}
$$

Then, we obtain deformed canonical commutation relation

$$
[x_i, p_j] = i\hbar(1 + (\frac{a}{\hbar})^2 p_i p_j)
$$

Lorentz Symmetry

ρ 表示に移ると、

$$S = \int dt d^d \rho \frac{1}{2} \left(\Phi(\rho,t)(\partial_a \partial^a - m^2)\Phi(\rho,t) \right).$$

すなわち ρ を回す『Lorentz 対称性』がある。
この表示を前提にして、Fermion の理論や、相互作用を作ることは可能。

$$S = \int dt \frac{d^d k}{(1+\lambda \boldsymbol{k}^2)^d} - i\bar{\Psi}(k,t)(\Gamma^0 \partial_t + i\Gamma^a k_i + m)\Psi(k,t),$$

$$\mathcal{D}\Psi = \prod_{\alpha,k,t} \frac{(1+\lambda \boldsymbol{k}^2)^d}{\Delta^d k} d\Psi^\alpha(k,t) d(\Psi^\alpha(k,t))^*.$$

Other canonical commutation relation

$$\begin{aligned}[\hat{x}^i, \hat{p}^j] &= f(\boldsymbol{p}^2)\delta^{ij}, \\ [\hat{x}^i, \hat{x}^j] &= 2f'(\boldsymbol{p}^2)(\hat{p}^i\hat{x}^j - \hat{p}^j\hat{x}^i).\end{aligned}$$

この場合も同様に、不変体積は

$$\frac{\prod_i \Delta x^i \Delta p^i}{f(\boldsymbol{p}^2)^d}.$$

よって、

$$\hat{\rho}^j \equiv i(f(\boldsymbol{p}^2))^{\frac{d}{2}}\partial_{k^j}(f(\boldsymbol{p}^2))^{-\frac{d}{2}}$$

で場の理論が作れる。

Other canonical commutation relation

よくある例は

$$f(\boldsymbol{p}^2) \quad = \quad (1+\lambda \boldsymbol{k}^2)^\alpha$$

$\alpha \geq \frac{1}{2}$ の場合にのみ、次元 d に依らない振舞いが見られる。すなわち $T^{1+d(1-2\alpha)}$ というものが、T^1 より leading にならないように。もしくは $\alpha = \frac{d-1}{2d}$ ととると、次元に依らず高温で T^2 という性質を持ってくれる。

$$\frac{d^d k}{(1+\lambda(\boldsymbol{k}^2+m^2))^d 2\omega} = \int d^{d+1} k \theta(k^0)(1+d)$$
$$\times \delta((1+\lambda(k^0)^2)^{d+1} - (1+\lambda(\boldsymbol{k}^2+m^2))^{d+1}).$$

という Lorentz 共変性の可能性も。