Riken, Saitama
2007-12-23

Developments in Integrability in Gauge/String Correspondence

Kazuhiro Sakai

(Keio University)

Based on collaborations with
N.Beisert, N.Gromov, V.Kazakov, Y.Satoh, P.Vieira, K.Zarembo

0. Introduction

AdS/CFT correspondence --- a gauge/string duality

(Maldacena '97, Gubser-Klebanov-Polyakov '97, Witten '97)

planar $\mathcal{N} = 4$ U(N) super Yang-Mills

$\Updownarrow$

- gluon dynamics common to QCDs

free IIB superstrings on $AdS_5 \times S^5$

- superstrings in the simplest curved background

- $N \to \infty$ limit $\Rightarrow$ integrability

Strong/weak correspondence

$\longrightarrow \lambda$

$\lambda \ll 1$ $\lambda \to \infty$

super Yang-Mills at one-loop $\Rightarrow$ higher loops

quantum strings $\Leftarrow$ classical strings on $AdS_5 \times S^5$

$\Downarrow$

conventional spin chain

integrable

(Minahan-Zarembo '02)
(Beisert-Staudacher '03)
(Beisert '03)

$\Downarrow$

sigma model on a coset space

integrable

(Bena-Roiban-Polchinski '03)

(expected to be) integrable at general λ

particle model

$(L \to \infty)$

(Staudacher '04)
(Beisert '05)

Plan of the talk

1. Super Yang-Mills at one-loop (spin chain)

2. Classical string theory (classical sigma model)

3. All-order SYM/quantum strings (particle model)

1. $\mathcal{N} = 4$ Super Yang-Mills

$\mathcal{N} = 4$ gauge multiplet

$\mathcal{N} = 2$ vector multiplet

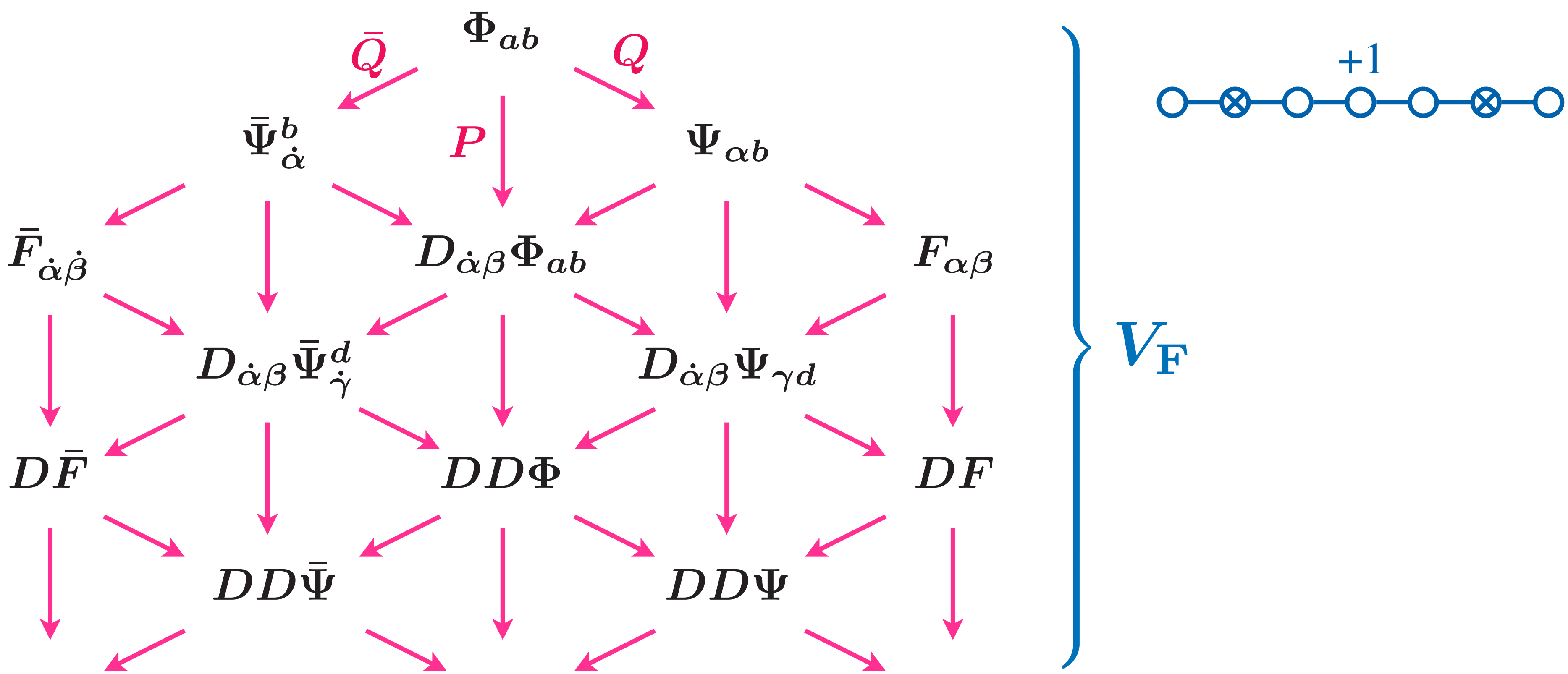
$\mathcal{N} = 2$ hypermultiplet

$\mathcal{N} = 1$ vector multiplet	$\mathcal{N} = 1$ chiral multiplet	$\mathcal{N} = 1$ chiral multiplet	$\mathcal{N} = 1$ chiral multiplet
A_μ Ψ_1	X Ψ_2	Y Ψ_3	Z Ψ_4
	$X = \Phi_1 + i\Phi_2$	$Y = \Phi_3 + i\Phi_4$	$Z = \Phi_5 + i\Phi_6$

$\mathcal{N} = 4$ Super Yang-Mills

$$\mathcal{L} = -\frac{1}{4g_{\mathrm{YM}}^2}\mathrm{Tr}\Big((F_{\mu\nu})^2 + 2(D_\mu\Phi_i)^2 - ([\Phi_i,\Phi_j])^2 + 2i\bar{\Psi}\not{D}\Psi - 2\bar{\Psi}\Gamma_i[\Phi_i,\Psi]\Big)$$

Global symmetry: $SO(4,2) \times SU(4) \subset PSU(2,2|4)$



Conformal Field Theory

- Correlation function of local operators

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \delta_{D_1 D_2}\frac{B_{12}}{|x_{12}|^{D_1+D_2}}$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{D_1+D_2-D_3}|x_{23}|^{D_2+D_3-D_1}|x_{31}|^{D_3+D_1-D_2}}$$

D_i : scaling dimension of the local operator $\mathcal{O}_i$

- Single trace operators

$$\mathcal{O} = \mathrm{Tr}[W_{A_1} W_{A_2} \cdots W_{A_J}]$$

$$W_A \in \{D^k \Phi, D^k \Psi, D^k \bar{\Psi}, D^k F\}$$

(Beisert '03)

  - dominant in the large N limit

- Scaling dimension:

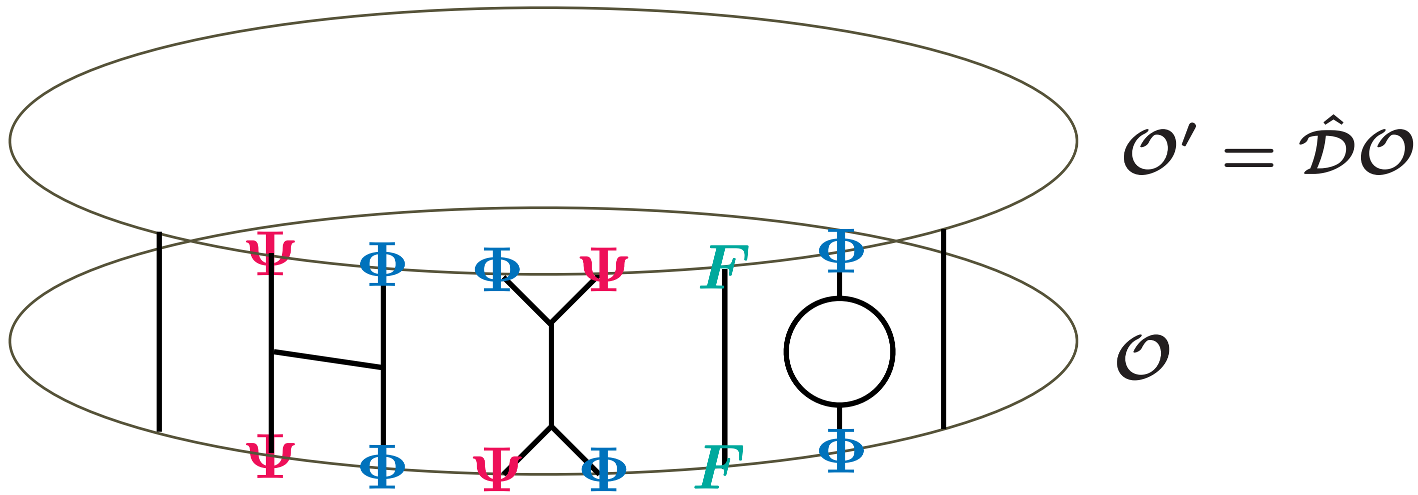
eigenvalue of the Dilatation operator $\hat{\mathcal{D}}$

- At tree level:

$$\hat{\mathcal{D}}_0 \mathcal{O} = \dim(\mathcal{O}) \mathcal{O}$$

$$[\Phi] = 1, \quad [\Psi] = \tfrac{3}{2}, \quad [F] = 2, \quad [D] = 1$$

- Quantum correction: operator mixing



$\hat{\mathcal{D}}$: non-diagonal matrix $\Longrightarrow$ spectral problem

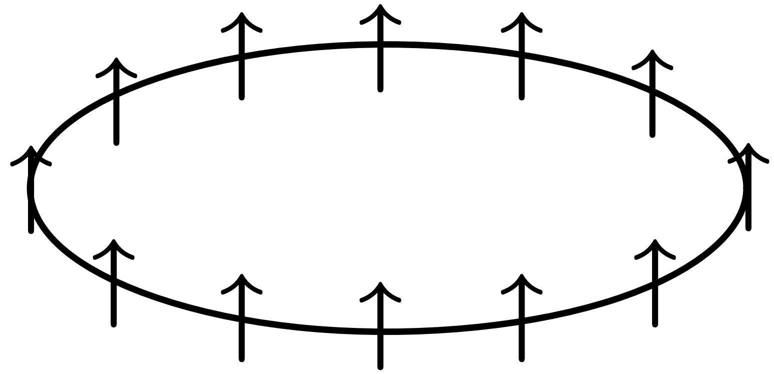
$$\hat{\mathcal{D}} = \sum_{n=0}^{\infty} \lambda^n \hat{\mathcal{D}}_n \qquad \lambda = g_{\mathrm{YM}}^2 N \quad \text{('t Hooft coupling)}$$

$\hat{\mathcal{D}}_1 \Leftrightarrow$ Hamiltonian of $\mathfrak{su}(2,2|4)$ spin chain

(Minahan-Zarembo '02) (Beisert-Staudacher '03)

- SU(2) subsector

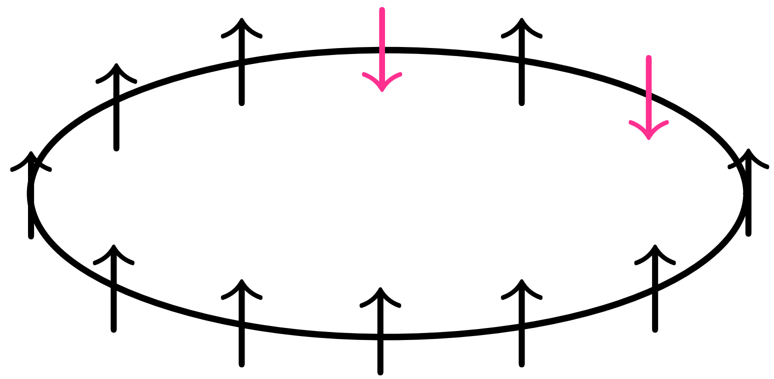
$$\mathrm{Tr}\, Z^L \quad \Leftrightarrow$$



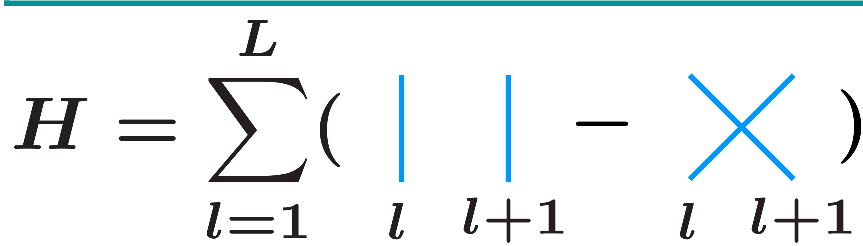
ferromagnetic vacuum

$$X = \Phi_1 + i\Phi_2$$
$$Z = \Phi_5 + i\Phi_6$$

$$\mathrm{Tr}(ZZZ\,X\,ZX\,ZZ\cdots) \quad \Leftrightarrow$$



XXX Heisenberg Spin chain



Bethe ansatz equation (coordinate Bethe ansatz)

One-magnon states

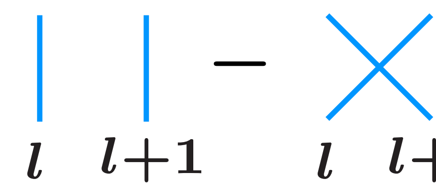
$$|\Psi(p)\rangle = \sum_{l=1}^{L} \psi(l) |\uparrow \cdots \uparrow \overset{l}{\downarrow} \uparrow \cdots \uparrow\rangle$$

$$\psi(l) = e^{ipl}$$

Schrödinger Eq.

$$H|\Psi\rangle = E|\Psi\rangle$$

$$H = \sum_{l=1}^{L} (\quad)$$



$$\begin{aligned} E &= 2 - e^{ip} - e^{-ip} \\ &= 4 \sin^2 \frac{p}{2} \end{aligned}$$

: Dispersion relation

Two-magnon states

$$|\Psi(p_1, p_2)\rangle = \sum_{1 \le l_1 < l_2 \le L} \psi(l_1, l_2) | \uparrow \cdots \uparrow \overset{l_1}{\downarrow} \uparrow \cdots \uparrow \overset{l_2}{\downarrow} \uparrow \cdots \uparrow \rangle$$

Schrödinger Eq. $\quad H|\Psi\rangle = E|\Psi\rangle$

$$E = \sum_{k=1}^{2} 4 \sin^2 \frac{p_k}{2}$$

(dispersion relation)

$$\psi(l_1, l_2) = e^{ip_1 l_1 + ip_2 l_2} + S(p_2, p_1) e^{ip_1 l_2 + ip_2 l_1}$$

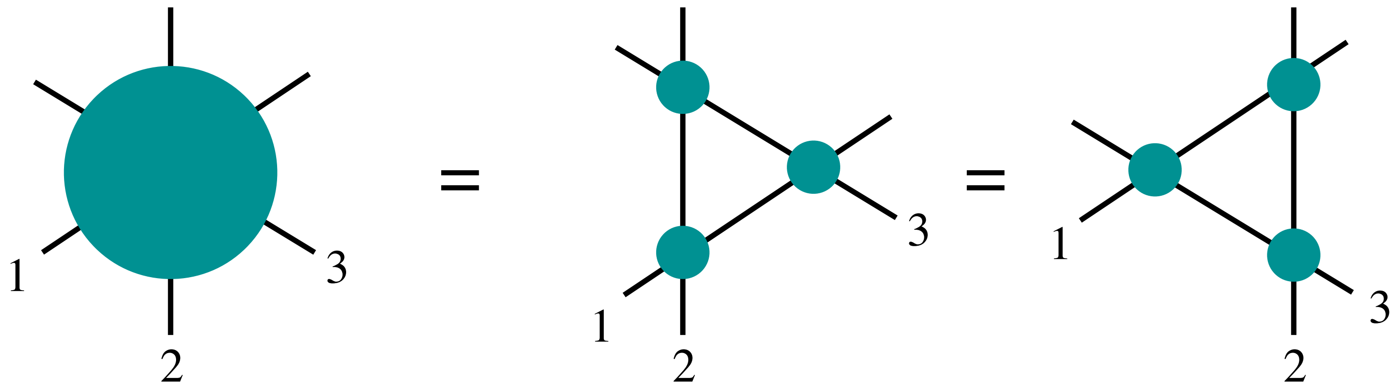
(Bethe's ansatz)

$$S(p_1, p_2) = -\frac{e^{ip_1 + ip_2} - e^{2ip_1} + 1}{e^{ip_1 + ip_2} - e^{2ip_2} + 1}$$

S-matrix

Integrability of 2D particle models

- Factorization of multi-particle scattering amplitudes



$$\left\{ \begin{array}{l} \text{dispersion relation } E(p) \text{ for 1 particle} \\ \text{scattering matrix } \hat{S}(p_1, p_2) \text{ for 2 particles} \end{array} \right.$$

$$\Rightarrow \left. \left\{ \begin{array}{l} \text{all scattering amplitudes} \\ \text{spectra of conserved charges} \end{array} \right\} \right. \text{are determined}$$

Factorized scattering

$$\psi(p_2, p_1, p_3, \ldots, p_J) = S(p_1, p_2)\psi(p_1, p_2, p_3, \ldots, p_J)$$

Periodic boundary condition

$$\psi(p_2, \ldots, p_J, p_1) = e^{-ip_1 L}\psi(p_1, \ldots, p_J)$$

Yang equations

$$e^{ip_k L} = \prod_{l \neq k}^{J} S(p_k, p_l)$$

$\Downarrow$ rapidity variable $\quad u = \frac{1}{2}\cot\frac{p}{2} \qquad \left(\frac{\partial p}{\partial u} = E\right)$

Bethe ansatz equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{l \neq k}^{J} \frac{u_k - u_l + i}{u_k - u_l - i} \qquad (k = 1, \ldots, J)$$

Local Charges

Momentum

$$P = Q_1 = \sum_k \frac{1}{i} \ln \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}$$

Energy

$$E = Q_2 = \sum_k \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}} \right)$$

Higher charges

$$Q_r = \sum_k \frac{1}{r-1} \left(\frac{i}{(u_k + \frac{i}{2})^{r-1}} - \frac{i}{(u_k - \frac{i}{2})^{r-1}} \right)$$

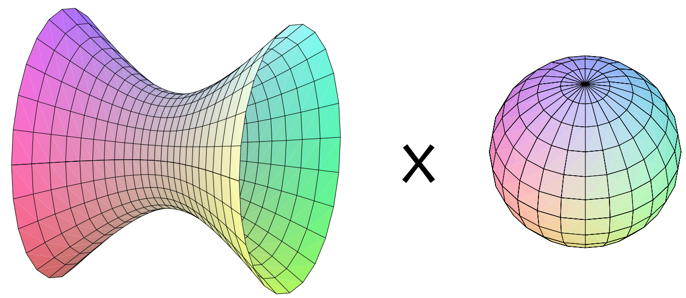
2. Classical strings

- AdS/CFT Correspondence

$\mathcal{N} = 4$ U(N)
Super Yang-Mills

$\Longleftrightarrow$

IIB Superstrings on



$AdS_5 \times S^5$

$$SO(4,2) \times SO(6) \subset PSU(2,2|4)$$

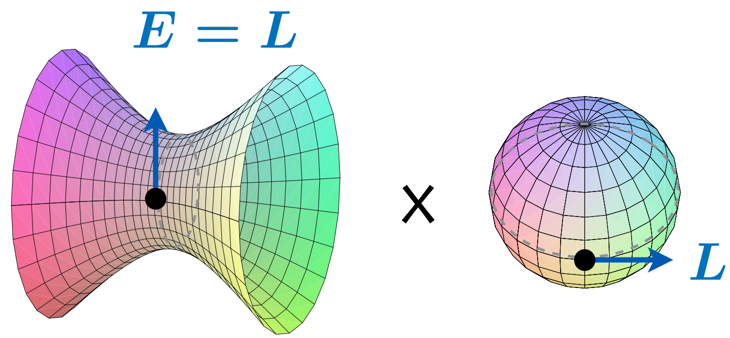
$$\lambda = g_{YM}^2 N \qquad\qquad R^4 = 4\pi g_s \alpha'^2 N$$

$$g_{YM}^2 = g_s$$

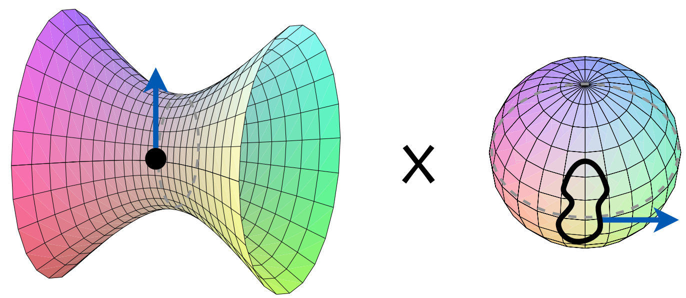
$$N \to \infty$$

$$4\pi\lambda = \frac{R^4}{\alpha'^2}$$

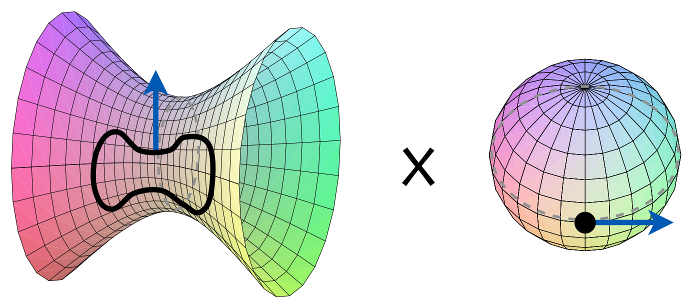
$$\mathcal{O} = \mathrm{Tr}(\overbrace{ZZZ\cdots ZZZ}^{L})$$



$$\mathcal{O} = \mathrm{Tr}(Z\cdots X\cdots \bar{Y}\cdots Z) + \cdots$$



$$\mathcal{O} = \mathrm{Tr}(Z\cdots \nabla^{s}Z\cdots \nabla^{s'}Z\cdots Z) + \cdots$$



Sigma model on $\mathbb{R}_t \times S^n$

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[-\partial_a X_0 \partial^a X_0 + \partial_a X_i \partial^a X_i + \Lambda \left(X_i X_i - 1 \right) \right]$$

$$(i = 1, \ldots, n)$$

Equations of motion

$$\partial_+ \partial_- X_i + (\partial_+ X_j \partial_- X_j) X_i = 0, \quad \partial_+ \partial_- X_0 = 0$$

Gauge: $X_0 = \kappa \tau$

$$\kappa = \frac{\Delta}{\sqrt{\lambda}}$$

Δ: energy of the string

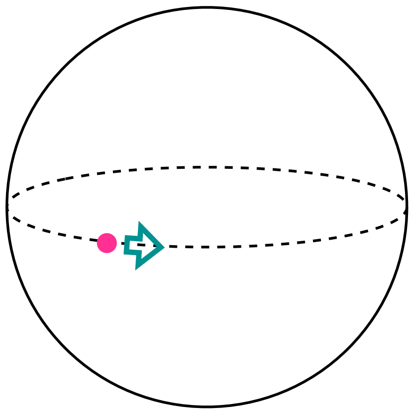
$$\left(\Delta = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \partial_\tau X_0 = \sqrt{\lambda} \kappa \right)$$

Virasoro constraints

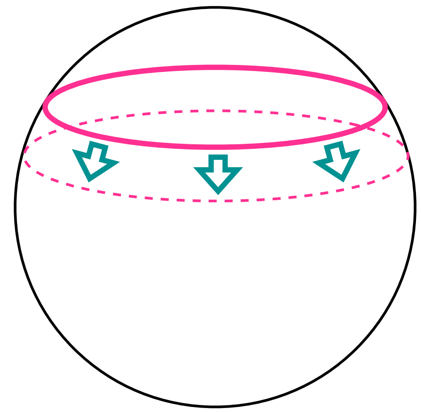
$$\begin{aligned} (\partial_\pm X_i)^2 &= (\partial_\pm X_0)^2 \\ &= \kappa^2 \end{aligned}$$

Examples of classical string solutions in $S^2 \times \mathbb{R}_t$

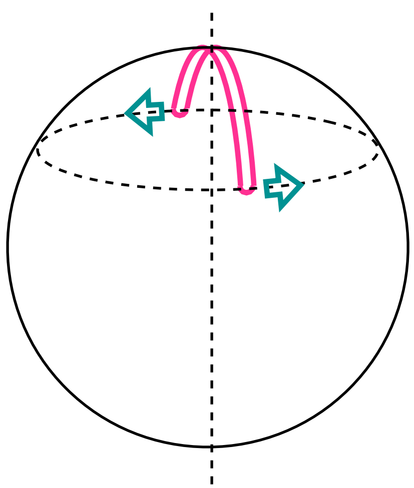
point-like string



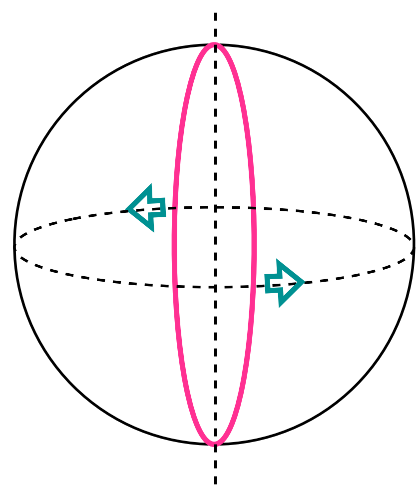
pulsating string

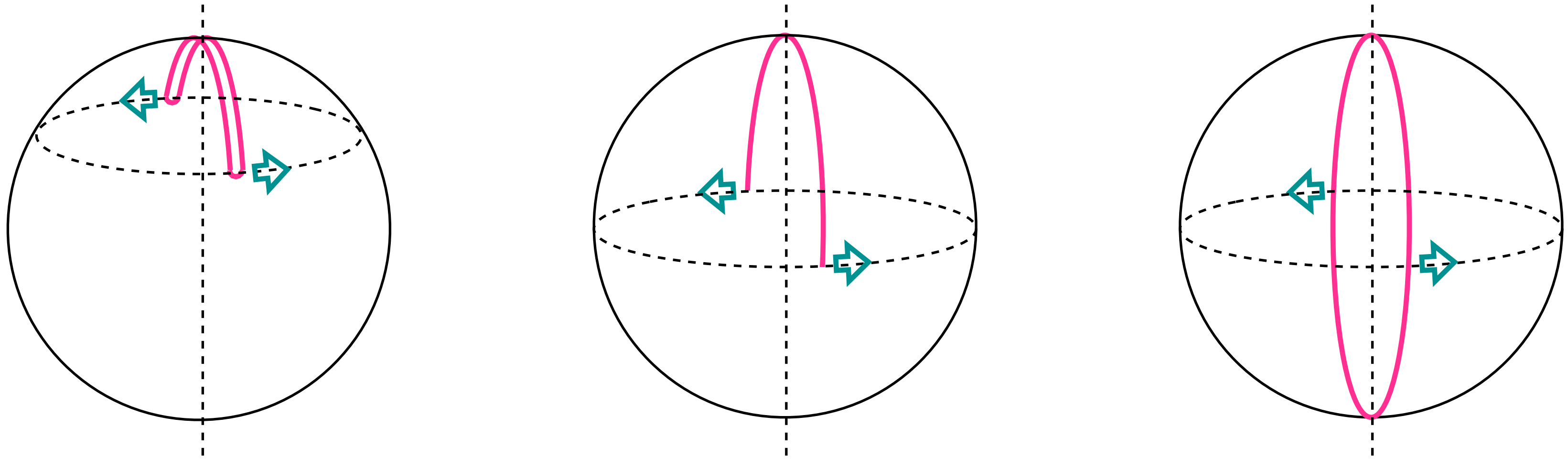


folded string



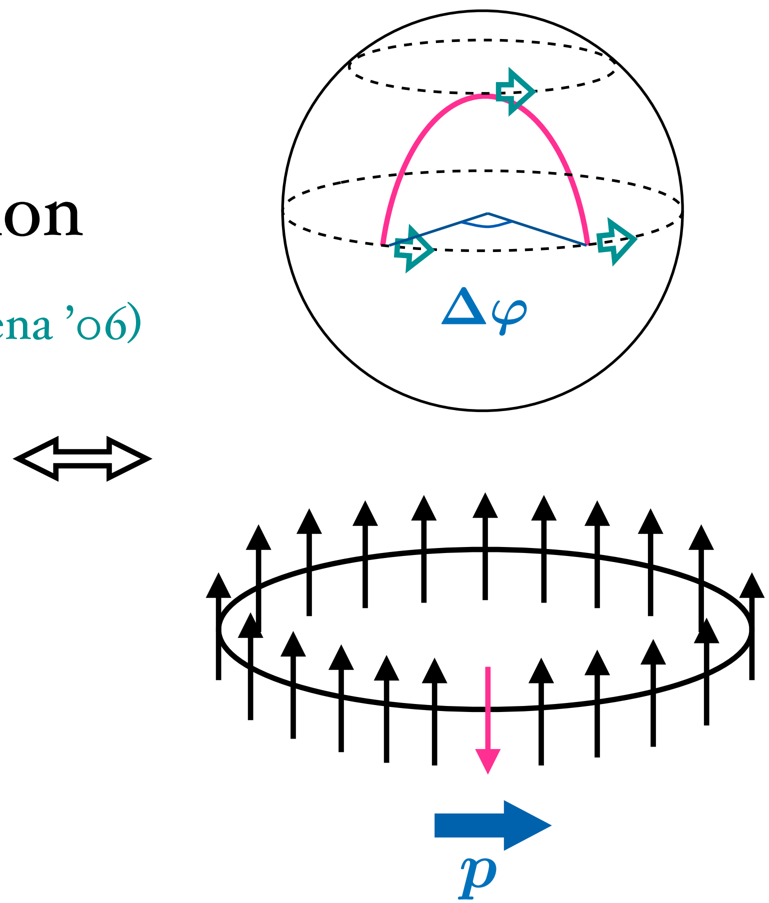
circular string



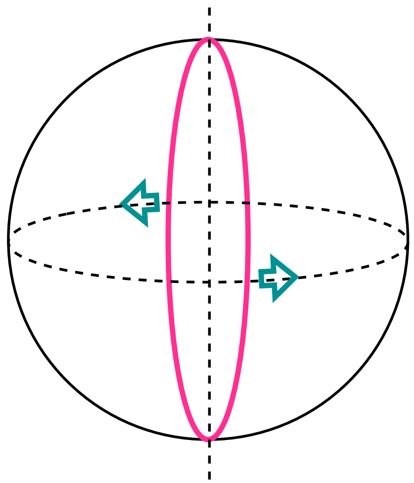


giant magnon

(Hofman-Maldacena '06)

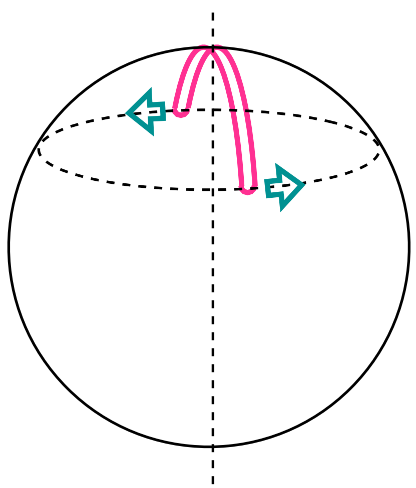


circular string



$$\vec{X} = \begin{pmatrix} \mathrm{sn}(\kappa\sigma|k)\cos\omega\tau \\ \mathrm{sn}(\kappa\sigma|k)\sin\omega\tau \\ \mathrm{cn}(\kappa\sigma|k) \end{pmatrix} \qquad k = \frac{\omega}{\kappa}$$

folded string



$$\vec{X} = \begin{pmatrix} k\,\mathrm{sn}(\omega\sigma|k)\cos\omega\tau \\ k\,\mathrm{sn}(\omega\sigma|k)\sin\omega\tau \\ \mathrm{dn}(\omega\sigma|k) \end{pmatrix} \qquad k = \frac{\kappa}{\omega}$$

Sigma model on $\mathbb{R}_t \times S^3$

$\cong$ SU(2) Principal Chiral Field Model

$$g \in \mathrm{SU}(2) \quad \leftrightarrow \quad \vec{X} \in S^3$$

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix}$$

Right current

$$j = -g^{-1} dg$$

$$dj - j \wedge j = 0, \qquad d * j = 0$$

Virasoro constraints

$$\frac{1}{2}\mathrm{Tr} j_{\pm}^2 = -\kappa^2$$

Lax Connection

$$a(x) = \frac{1}{1-x^2} j + \frac{x}{1-x^2} * j$$

x : spectral parameter

$$dj - j \wedge j = 0$$
$$d * j = 0$$

$\Leftrightarrow$

$$d\,a(x) - a(x) \wedge a(x) = 0$$

$\Leftrightarrow$

$$[\mathcal{L}(x), \mathcal{M}(x)] = 0$$

Lax pair

$$\mathcal{L}(x) = \partial_\sigma - a_\sigma(x) = \partial_\sigma - \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right)$$
$$\mathcal{M}(x) = \partial_\tau - a_\tau(x) = \partial_\tau - \frac{1}{2}\left(\frac{j_+}{1-x} + \frac{j_-}{1+x}\right)$$

Auxiliary Linear Problem

$$\begin{cases} \mathcal{L}(x)\Psi(x;\tau,\sigma) = 0 \\ \mathcal{M}(x)\Psi(x;\tau,\sigma) = 0 \end{cases} \qquad \begin{cases} \partial_\sigma \Psi = a_\sigma \Psi \\ \partial_\tau \Psi = a_\tau \Psi \end{cases}$$

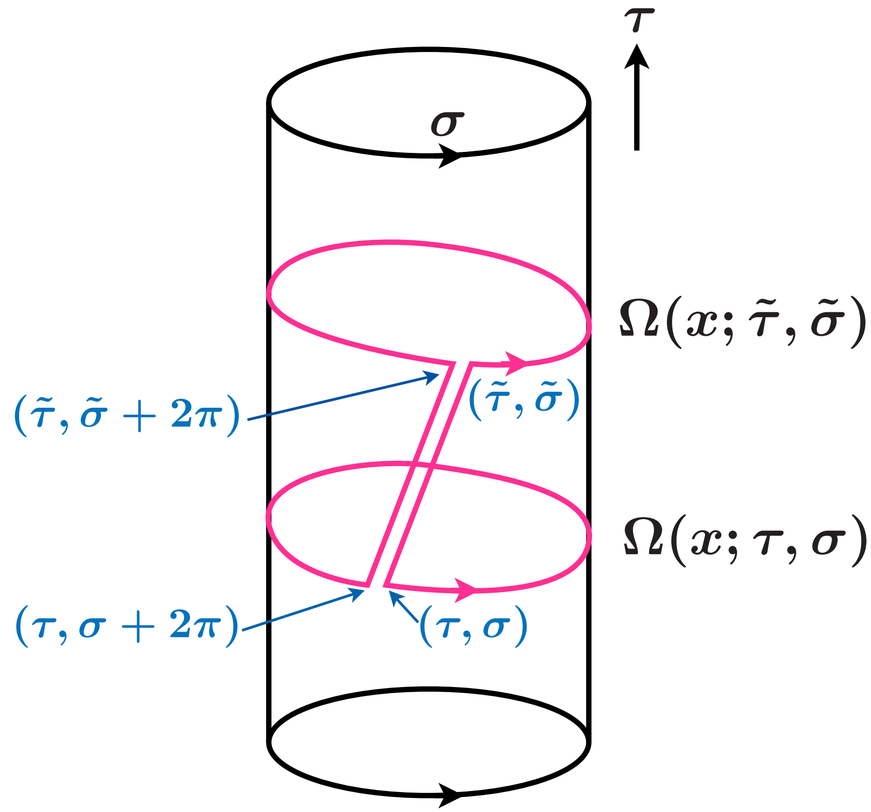
$$\Psi(x;\tau,\sigma) = \mathrm{P}\exp\int_0^\sigma a_\sigma d\sigma$$

Monodromy matrix

$$\Psi(x;\tau,\sigma+2\pi) = \Omega(x;\tau,\sigma)\Psi(x;\tau,\sigma)$$

$$\Omega(x;\tau,\sigma) = \mathrm{P}\exp\int_0^{2\pi} a_\sigma d\sigma$$

Monodromy matrix



$$\Omega(x;\tilde\tau,\tilde\sigma) = U^{-1}\Omega(x;\tau,\sigma)U$$

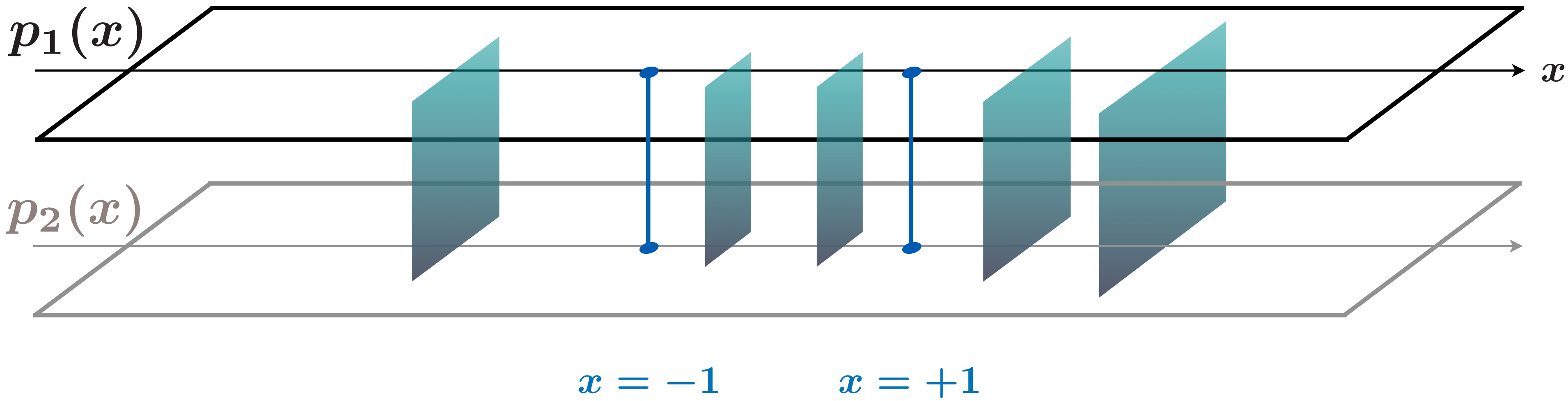
$$\Omega(x) \sim \begin{pmatrix} e^{ip_1(x)} & 0 \\ 0 & e^{ip_2(x)} \end{pmatrix}$$

$$p_1(x) = -p_2(x) =: p(x)$$

quasi-momentum

Spectral curve

(Kazakov-Marshakov-Minahan-Zarembo '04)



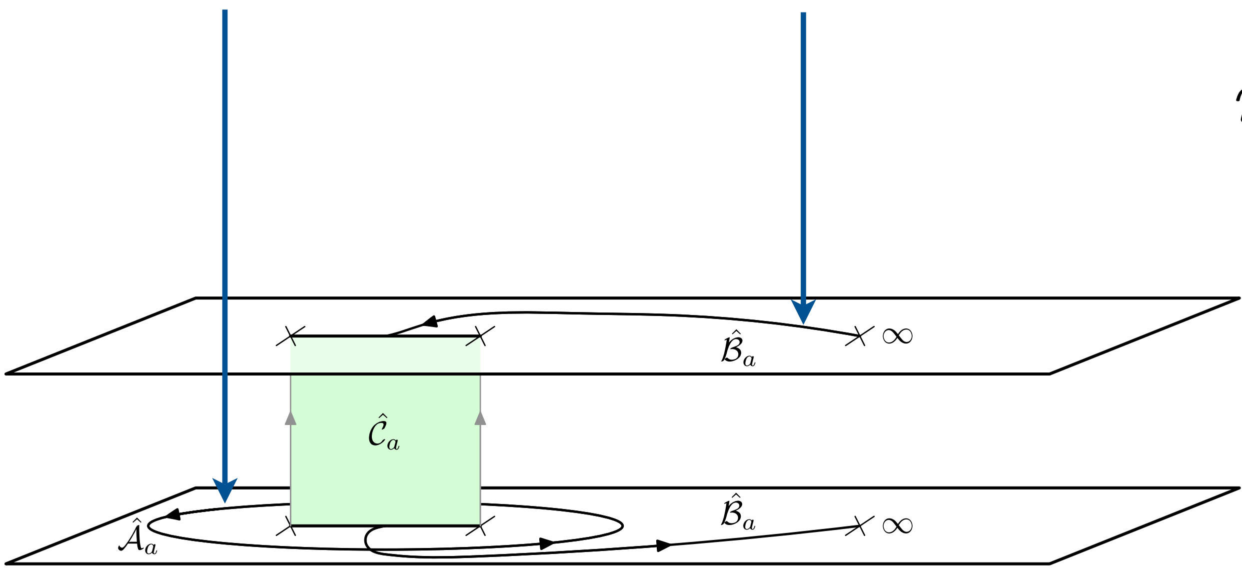
- Virasoro Constraints

$$\frac{1}{2}\mathrm{Tr}j_{\pm}^2 = -\kappa^2 \quad \Rightarrow \quad p(x) \sim -\frac{\pi\kappa}{x \mp 1} \qquad (x \to \pm 1)$$

- Branch choice

$$\oint_{\hat{\mathcal{A}}_a} dp = 0, \qquad \int_{\hat{\mathcal{B}}_a} dp = 2\pi \hat{n}_a$$

$\hat{n}_a$: mode number



Explicit form of general finite gap solution

(Dorey-Vicedo '06)

$$X_1 + iX_2 = C_1 \frac{\theta\left(2\pi \int_{\infty^+}^{0^+} \vec{\omega} - \oint_{\vec{b}} d\mathcal{Q} - \vec{D}\right)}{\theta\left(\oint_{\vec{b}} d\mathcal{Q} + \vec{D}\right)} \exp\left(-i \int_{\infty^+}^{0^+} d\mathcal{Q}\right)$$

$$X_3 + iX_4 = C_2 \frac{\theta\left(2\pi \int_{\infty^-}^{0^+} \vec{\omega} - \oint_{\vec{b}} d\mathcal{Q} - \vec{D}\right)}{\theta\left(\oint_{\vec{b}} d\mathcal{Q} + \vec{D}\right)} \exp\left(-i \int_{\infty^-}^{0^+} d\mathcal{Q}\right)$$

$$\theta(\vec{z}) = \sum_{\vec{m} \in \mathbb{Z}^g} \exp\left(i\vec{m} \cdot \vec{z} + \pi i (\Pi \vec{m}) \cdot \vec{m}\right)$$: Riemann theta function

$d\mathcal{Q} = \sigma dp + \tau dq$ p : quasi-momentum q : quasi-energy

ω_j : normalized holomorphic differentials $\left(\oint_{\mathcal{A}_i} \omega_j = \delta_{ij}\right)$

$b_j = \mathcal{B}_j - \mathcal{B}_{g+1}$: closed B-cycles

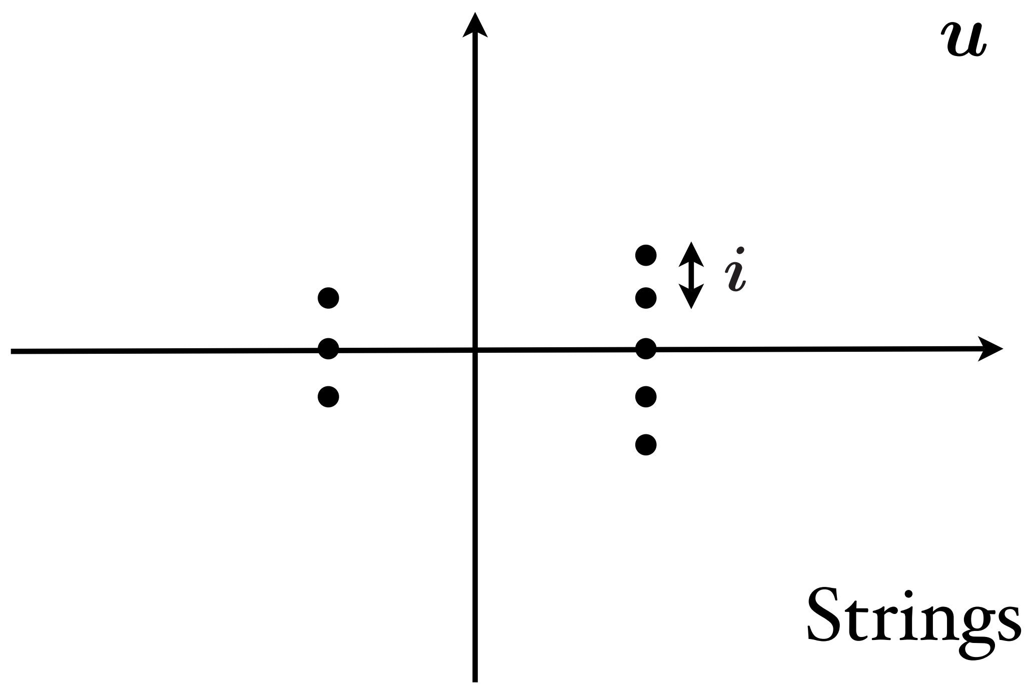
$\vec{D}, C_1, C_2$: constants

Finite gap solution on the Yang-Mills side

- Thermodynamic limit

$$\left(\frac{u_p + \frac{i}{2}}{u_p - \frac{i}{2}}\right)^L = \prod_{\substack{q=1 \\ q \neq p}}^{J} \frac{u_p - u_q + i}{u_p - u_q - i}$$

$$L \to \infty, \quad u_k \sim O(1)$$



- Thermodynamic limit with rescaling of rapidities

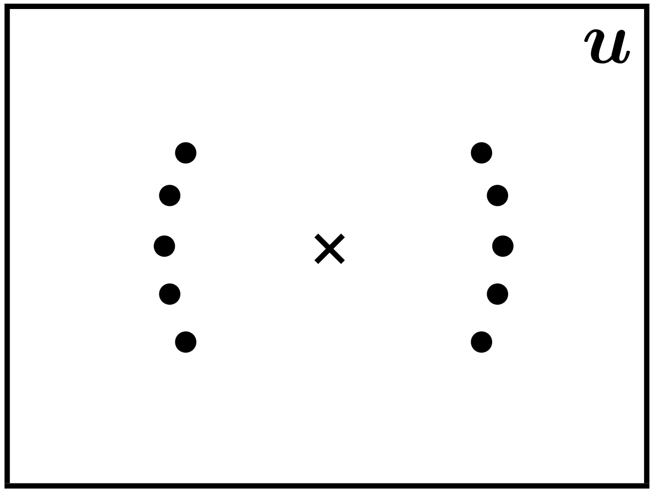
$$\left(\frac{u_p + \frac{i}{2}}{u_p - \frac{i}{2}}\right)^L = \prod_{\substack{q=1 \\ q\neq p}}^{J} \frac{u_p - u_q + i}{u_p - u_q - i}$$

$$L, J \to \infty, \qquad u_k \to L u_k$$

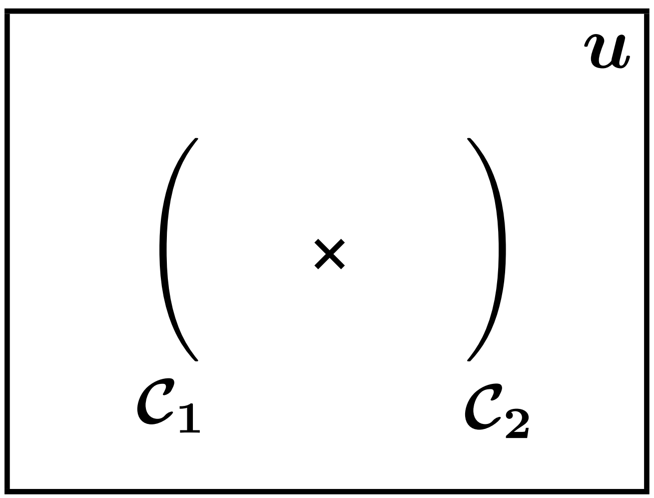
Log of both sides

$$\frac{1}{u_p} + 2\pi n_p = \frac{2}{L} \sum_{q\neq p}^{J} \frac{1}{u_p - u_q}$$

$$n_p \in \mathbb{Z} \qquad \text{: mode number}$$



$\Downarrow$



Resolvent

$$G(u) = \frac{1}{L}\sum_{q=1}^{J}\frac{1}{u-u_q}$$

$$\Downarrow$$

$$G(u) = \int_{\mathcal{C}} \frac{dv\,\rho(v)}{u-v}$$

BAE

$$\frac{1}{u} + 2\pi n_a = 2\not{G}(u)$$

for $u \in \mathcal{C}_a$

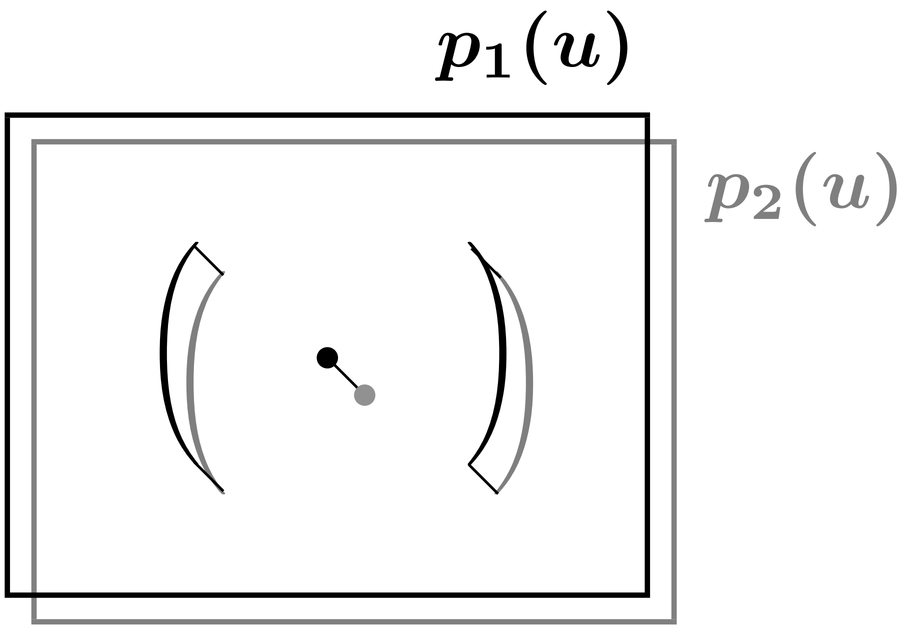
Quasi-momenta

$$p_1(u) = -p_2(u) = G(u) - \frac{1}{2u}$$

BAE

$$\frac{1}{u} + 2\pi n_a = 2\not{G}(u)$$

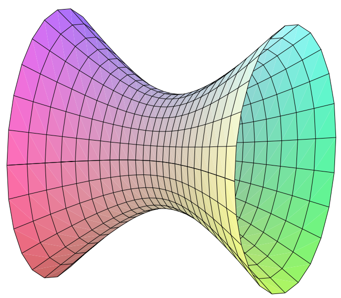
$$\Leftrightarrow \quad \boxed{p_1(u + i0) = p_2(u - i0) + 2\pi n_a \quad (u \in \mathcal{C}_a)}$$



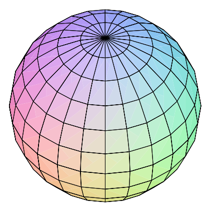
hyper-elliptic curve

Classical Superstring on $AdS_5 \times S^5$

Classical IIB Superstrings
on



$\times$



$\boldsymbol{AdS_5}$ $\boldsymbol{S^5}$

$=$

Coset sigma-model
on

$$\frac{\mathbf{PSU(2,2|4)}}{\mathbf{Sp(1,1)\times Sp(2)}}$$

$$\begin{matrix} \boldsymbol{X^i(\sigma,\tau)} \\ \boldsymbol{\psi_\alpha(\sigma,\tau)} \end{matrix} \quad \longrightarrow \quad \boldsymbol{g(\sigma,\tau)} \in \mathbf{PSU(2,2|4)}$$

$$\boldsymbol{J = -g^{-1}dg}$$

decomposition w.r.t. $\mathbb{Z}_4$-grading

$$\boldsymbol{J = H + Q_1 + P + Q_2}$$

Sigma-Model Action

(Metsaev-Tseytlin '98)
(Roiban-Siegel '02)

$$S_\sigma = \frac{\sqrt{\lambda}}{2\pi}\int \left(\tfrac{1}{2}\mathrm{str}P \wedge *P - \tfrac{1}{2}\mathrm{str}Q_1 \wedge Q_2 + \Lambda \wedge \mathrm{str}P\right)$$

Lax Connection

(Bena-Polchinski-Roiban '03)

$$\begin{aligned} A(z) \;=\; & H + \left(\tfrac{1}{2}z^2 + \tfrac{1}{2}z^{-2}\right) P \\ & + \left(-\tfrac{1}{2}z^2 + \tfrac{1}{2}z^{-2}\right)(*P - \Lambda) + z^{-1}Q_1 + z\,Q_2 \end{aligned}$$

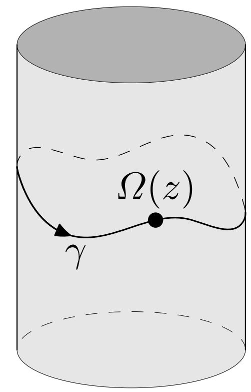
$\left\{\begin{array}{l}\text{Bianchi Identity} \qquad dJ - J \wedge J = 0 \\ \text{Equation of Motion}\end{array}\right.$

$\Leftrightarrow$ Flatness Condition

$$dA(z) - A(z) \wedge A(z) = 0$$

Monodromy Matrix

$$\Omega(z) = \frac{\mathbf{P}\exp\int_0^{2\pi} d\sigma\, A(z)}{\mathbf{P}\exp\int_0^{2\pi} d\sigma\, A(1)}$$



Physical quantity: Conjugacy class of $\Omega(z)$

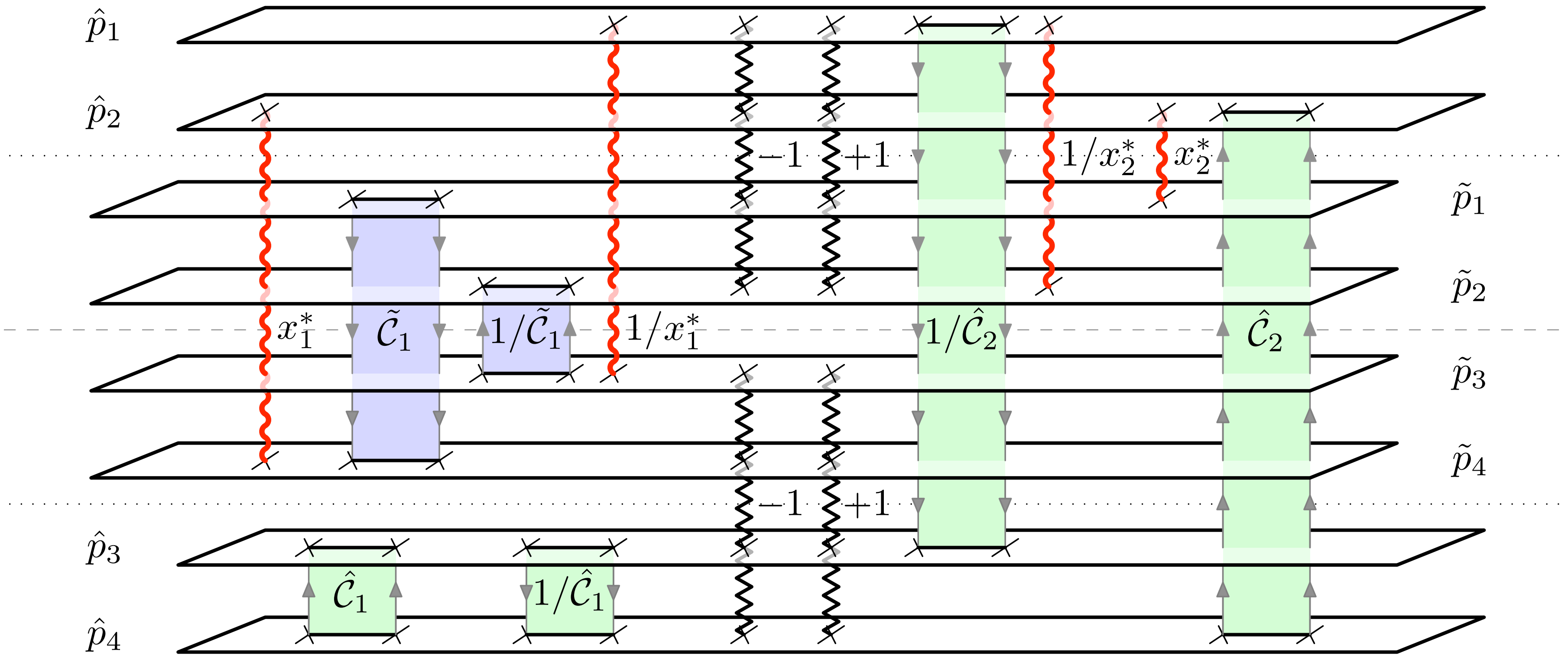
($\Rightarrow$ Generating functions of conserved charges)

Eigenvalues of the Monodromy Matrix

$$\begin{aligned}\Omega^{\mathrm{diag}}(z) &= u(z)\Omega(z)u(z)^{-1} \\ &= \mathrm{diag}(e^{i\tilde{p}_1}, e^{i\tilde{p}_2}, e^{i\tilde{p}_3}, e^{i\tilde{p}_4} | e^{i\hat{p}_1}, e^{i\hat{p}_2}, e^{i\hat{p}_3}, e^{i\hat{p}_4})\end{aligned}$$

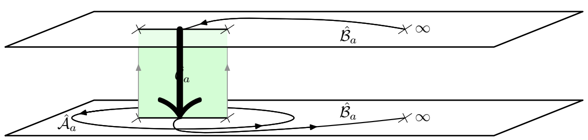
$\tilde{p}_i(z), \hat{p}_i(z)$: quasi-momenta

Spectral curve for a classical string solution

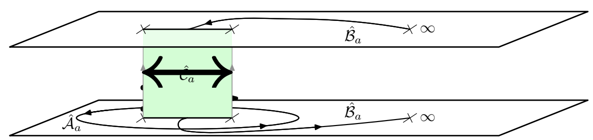


(1) distribution of cuts (Beisert-Kazakov-K.S.-Zarembo '05)

(2) mode numbers



(3) fillings



Conserved Charges

- Angular Momenta

$$J_2 - J_3 = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx(\tilde{p}_1(x) - \tilde{p}_2(x))$$

$$J_1 - J_2 = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx(\tilde{p}_2(x) - \tilde{p}_3(x))$$

$$J_2 + J_3 = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_\infty dx(\tilde{p}_3(x) - \tilde{p}_4(x))$$

- Energy

$$\delta E = -\frac{\sqrt{\lambda}}{4\pi^2 i} \sum_{a=1}^{A} \oint_{\mathcal{A}_a} \frac{dx}{x^2} (\tilde{p}_1(x) + \tilde{p}_2(x) - \hat{p}_1(x) - \hat{p}_2(x))$$

3. The particle model

(Staudacher '04) (Beisert '05, '06)

- Vacuum

$$|\mathbf{0}\rangle^{\mathbf{I}} := |\cdots ZZZZZZZZZZZZZZZZZZ\cdots\rangle$$

- Asymptotic state

$$|\boldsymbol{X}_1\boldsymbol{X}_2\rangle^{\mathbf{I}}$$

$$:= \sum_{n_1<n_2} e^{ip_1 n_1 + ip_2 n_2}|\cdots ZZZ\boldsymbol{X}_1 ZZZ\cdots ZZZ\boldsymbol{X}_2 ZZZ\cdots\rangle$$

- One particle states: 8 bosons + 8 fermions

(Berenstein-Maldacena-Nastase '02)

$$X, Y, \bar{X}, \bar{Y}, \quad D_i Z \ {}_{(i=1,\dots,4)},$$
$$\Psi_{\alpha\dot{a}}, \Psi_{a\dot{\alpha}} \ {}_{(a,\alpha=1,\dots,2)}$$

: single excitation of Z

$\bar{Z}, F_{\alpha\beta}, D_i\Phi_j, \dots$: multiple excitation

- Spontaneous breaking of the global symmetry

$$\begin{array}{ccccccc} PSU(2,2|4) & \rightarrow & PSU(2|2) & \times & PSU(2|2) & \ltimes \mathbb{R} \\ (8|8) & = & (2|2) & \times & (2|2) & \end{array}$$

Z	ϕ_1	ϕ_2	ψ_1	ψ_2
$\bar{\phi}_1$	X	Y	Ψ_{11}	Ψ_{12}
$\bar{\phi}_2$	$\bar{Y}$	$\bar{X}$	Ψ_{21}	Ψ_{22}
$\bar{\psi}_1$	$\dot{\Psi}_{11}$	$\dot{\Psi}_{12}$	$D_{11}Z$	$D_{12}Z$
$\bar{\psi}_2$	$\dot{\Psi}_{21}$	$\dot{\Psi}_{22}$	$D_{21}Z$	$D_{22}Z$

- centrally extended $\mathfrak{su}(2|2)$ algebra

$$
\begin{aligned}
[R^a{}_b, J^c] &= \delta^c_b J^a - \tfrac{1}{2}\delta^a_b J^c \\
[L^\alpha{}_\beta, J^\gamma] &= \delta^\gamma_\beta J^\alpha - \tfrac{1}{2}\delta^\alpha_\beta J^\gamma \\
\{Q^\alpha{}_a, S^b{}_\beta\} &= \delta^b_a L^\alpha{}_\beta + \delta^\alpha_\beta R^b{}_a + \delta^b_a \delta^\alpha_\beta C \\
\{Q^\alpha{}_a, Q^\beta{}_b\} &= \epsilon^{\alpha\beta}\epsilon_{ab} P \\
\{S^a{}_\alpha, S^b{}_\beta\} &= \epsilon^{ab}\epsilon_{\alpha\beta} K
\end{aligned}
$$

- transformation of the one-particle states

$$
\begin{aligned}
Q^\alpha{}_a |\phi^b\rangle^{\mathrm{I}} &= a\, \delta^b_a |\psi^\alpha\rangle^{\mathrm{I}} \\
Q^\alpha{}_a |\psi^\beta\rangle^{\mathrm{I}} &= b\, \epsilon^{\alpha\beta}\epsilon_{ab} |\phi^b Z^+\rangle^{\mathrm{I}} \\
S^a{}_\alpha |\phi^b\rangle^{\mathrm{I}} &= c\, \epsilon^{ab}\epsilon_{\alpha\beta} |\psi^\beta Z^-\rangle^{\mathrm{I}} \\
S^a{}_\alpha |\psi^\beta\rangle^{\mathrm{I}} &= d\, \delta^\beta_\alpha |\phi^a\rangle^{\mathrm{I}}
\end{aligned}
$$

$$|X\rangle^{\mathrm{I}} = \sum_n e^{ipn} |\cdots ZZ\overset{\overset{n}{\downarrow}}{X}ZZ\cdots\rangle$$

$$|Z^+X\rangle^{\mathrm{I}} = \sum_n e^{ipn} |\cdots ZZZ\overset{\overset{n+1}{\downarrow}}{X}ZZ\cdots\rangle$$

$$|XZ^+\rangle^{\mathrm{I}} = \sum_n e^{ipn} |\cdots ZZ\overset{\overset{n}{\downarrow}}{X}ZZZ\cdots\rangle$$

$$|Z^{\pm}X\rangle^{\mathrm{I}} = e^{\mp ip} |XZ^{\pm}\rangle^{\mathrm{I}}$$

$$C^2 - PK = \frac{1}{4} \quad \Longrightarrow \quad E = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)} - 1$$

Casimir invariant
(for the 4-dim rep.)

dispersion relation

(Beisert-Dippel-Staudacher '04)

$$\left(C = \frac{1}{2} n_{\text{particle}} + \frac{1}{2} E\right)$$

$\mathfrak{su}(2|2)$ S-matrix $\mathcal{S}(p_1, p_2; \lambda)$

(Beisert '05, '06)

- 2-body scattering matrix of 4-dim reps. (16×16 matrix)

- invariant under the centrally extended $\mathfrak{su}(2|2)$

  $\Longrightarrow$ fully determined up to an overall scalar

- It satisfies unitarity and Yang-Baxter eqs.

- Equivalent up to a similarity transf. with the Shastry's R-matrix ($\rightarrow$ integrability of the Hubbard model)

  (not of the difference form) $\mathcal{S}_{12}(~~u_1 - u_2~~)$

$$
\begin{aligned}
S(p_1,p_2) = & \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\Big(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\Big) \\
& + \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\Big(E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1 - E_1^2\otimes E_2^1 - E_2^1\otimes E_1^2\Big) \\
& - \Big(E_3^3\otimes E_3^3 + E_4^4\otimes E_4^4 + E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3\Big) \\
& + \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\Big(E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3 - E_3^4\otimes E_4^3 - E_4^3\otimes E_3^4\Big) \\
& + \frac{x_2^- - x_1^-}{x_2^+ - x_1^-}\frac{\eta_1}{\tilde\eta_1}\Big(E_1^1\otimes E_3^3 + E_1^1\otimes E_4^4 + E_2^2\otimes E_3^3 + E_2^2\otimes E_4^4\Big) \\
& + \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\frac{\eta_2}{\tilde\eta_2}\Big(E_3^3\otimes E_1^1 + E_4^4\otimes E_1^1 + E_3^3\otimes E_2^2 + E_4^4\otimes E_2^2\Big) \\
& + i\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde\eta_1\tilde\eta_2}\Big(E_1^4\otimes E_2^3 + E_2^3\otimes E_1^4 - E_2^4\otimes E_1^3 - E_1^3\otimes E_2^4\Big) \\
& + i\frac{x_1^- x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+(x_1^- - x_2^+)(1 - x_1^- x_2^-)}\Big(E_3^2\otimes E_4^1 + E_4^1\otimes E_3^2 - E_4^2\otimes E_3^1 - E_3^1\otimes E_4^2\Big) \\
& + \frac{x_1^+ - x_1^-}{x_1^- - x_2^+}\frac{\eta_2}{\tilde\eta_1}\Big(E_1^3\otimes E_3^1 + E_1^4\otimes E_4^1 + E_2^3\otimes E_3^2 + E_2^4\otimes E_4^2\Big) \\
& + \frac{x_2^+ - x_2^-}{x_1^- - x_2^+}\frac{\eta_1}{\tilde\eta_2}\Big(E_3^1\otimes E_1^3 + E_4^1\otimes E_1^4 + E_3^2\otimes E_2^3 + E_4^2\otimes E_2^4\Big)
\end{aligned}
$$

(For notations, see: hep-th/0612229 by Arutyunov-Frolov-Zamaklar '06)

SPIN CHAIN BASIS: $\eta_1 = \eta(p_1)\,,\quad \eta_2 = \eta(p_2)\,,\quad \tilde\eta_1 = \eta(p_1)\,,\quad \tilde\eta_2 = \eta(p_2)$

STRING BASIS: $\eta_1 = \eta(p_1)e^{\frac{i}{2}p_2}\,,\quad \eta_2 = \eta(p_2)\,,\quad \tilde\eta_1 = \eta(p_1)\,,\quad \tilde\eta_2 = \eta(p_2)e^{\frac{i}{2}p_1}$

$$\eta(p) = \sqrt{ix^-(p) - ix^+(p)} \qquad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\,, \qquad \frac{x^+}{x^-} = e^{ip}$$

The S-matrix is concisely expressed in terms of new rapidity variables $x^{\pm}$

$$x^{\pm}(u) = x(u \pm \tfrac{i}{2})$$

$$x(u) = \frac{u}{2}\left(1 + \sqrt{1 - 4g^2/u^2}\right) \qquad \left(g = \frac{\sqrt{\lambda}}{4\pi}\right)$$

$$\Longleftrightarrow$$

$$u \pm \tfrac{i}{2} = x^{\pm} + \frac{g^2}{x^{\pm}}$$

Full $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ S-matrix

(dressing phase)

$$\hat{S} = S_0^2 [\hat{R}_{\mathfrak{su}(2|2)} \otimes \hat{R}_{\mathfrak{su}(2|2)}]$$

$$S_0(p_k, p_j)^2 = \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} e^{2i\theta(u_k, u_j)}$$

- Periodic boundary condition

Yang equations: $$e^{ip_k L} = \prod_{j \neq k}^{K_4} \hat{S}(p_k, p_j)$$

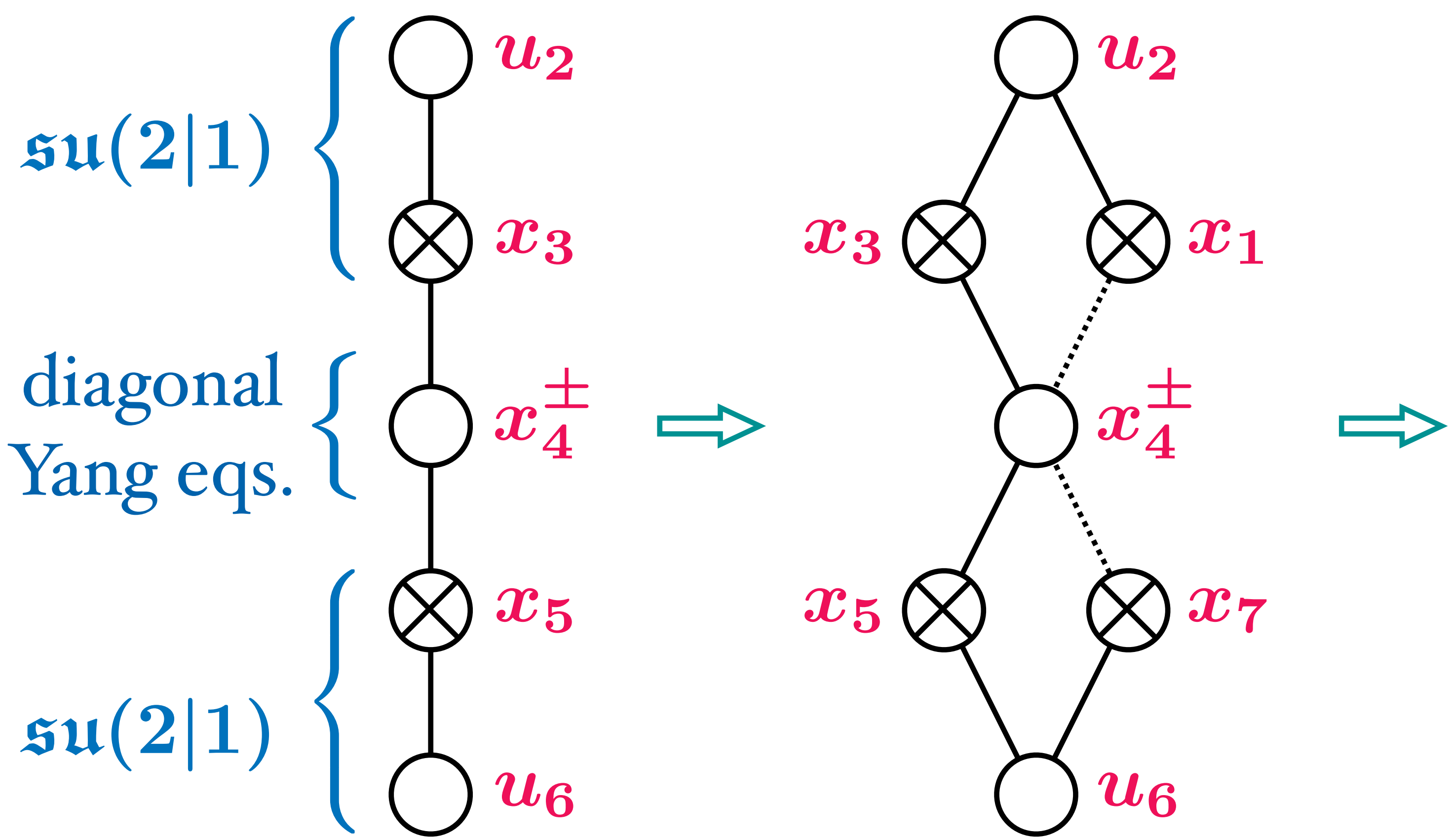
$\Downarrow$ diagonalization (by nested Bethe ansatz)

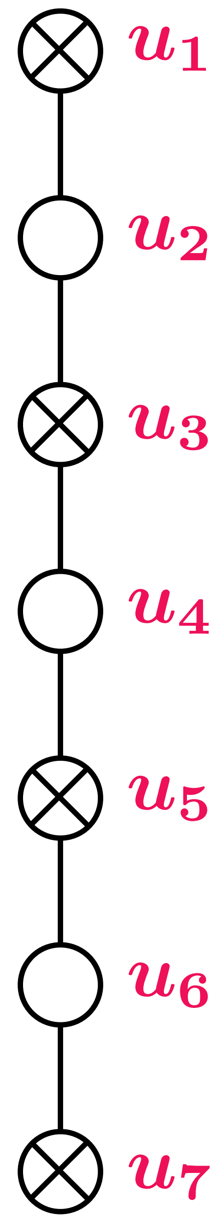
(Beisert '05, Martins-Melo '07, de Leeuw '07, ...)

Asymptotic all-order $\mathfrak{psu}(2,2|4)$ Bethe equations diagonalization of Yang eqs.

Beisert-Staudacher all-loop Bethe eqs.

one-loop





$$x_3 \longrightarrow g^2/x_1$$
$$x_5 \longrightarrow g^2/x_7$$

$$g \longrightarrow 0$$

All-order Bethe equations (Beisert-Staudacher '05)

$$1 = \prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + i/2}{u_{1,k} - u_{2,j} - i/2} \prod_{j=1}^{K_4} \frac{1 - g^2/x_{1,k}\, x_{4,j}^+}{1 - g^2/x_{1,k}\, x_{4,j}^-},$$

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + i/2}{u_{2,k} - u_{3,j} - i/2} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + i/2}{u_{2,k} - u_{1,j} - i/2},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + i/2}{u_{3,k} - u_{2,j} - i/2} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-},$$

$$\left(\frac{x_{4,k}^+}{x_{4,k}^-}\right)^J = \prod_{j\neq k}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i}\, \underline{e^{2i\theta(u_{4,k},u_{4,j})}} \prod_{j=1}^{K_1} \frac{1 - g^2/x_{4,k}^-\, x_{1,j}}{1 - g^2/x_{4,k}^+\, x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}}$$

(dressing phase)

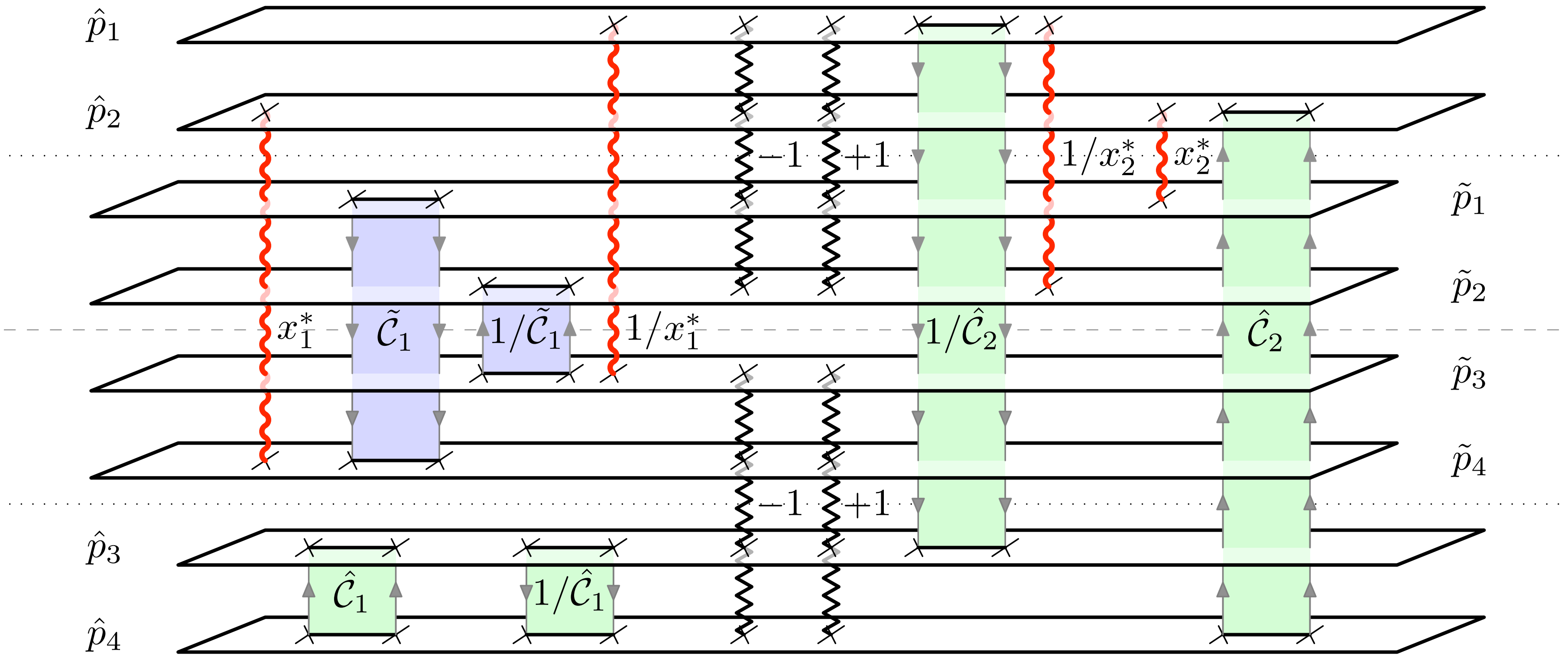
$$\times \prod_{j=1}^{K_7} \frac{1 - g^2/x_{4,k}^-\, x_{7,j}}{1 - g^2/x_{4,k}^+\, x_{7,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + i/2}{u_{5,k} - u_{6,j} - i/2} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^+}{x_{5,k} - x_{4,j}^-},$$

$$1 = \prod_{j\neq k}^{K_6} \frac{u_{6,k} - u_{6,j} - i}{u_{6,k} - u_{6,j} + i} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + i/2}{u_{6,k} - u_{5,j} - i/2} \prod_{j=1}^{K_7} \frac{u_{6,k} - u_{7,j} + i/2}{u_{6,k} - u_{7,j} - i/2},$$

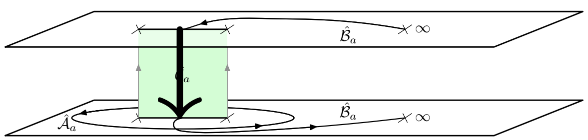
$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + i/2}{u_{7,k} - u_{6,j} - i/2} \prod_{j=1}^{K_4} \frac{1 - g^2/x_{7,k}\, x_{4,j}^+}{1 - g^2/x_{7,k}\, x_{4,j}^-}.$$

Spectral curve for a classical string solution

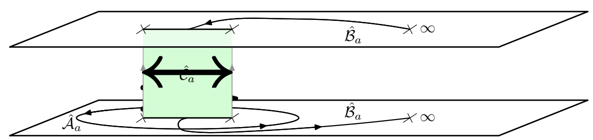


(1) distribution of cuts (Beisert-Kazakov-K.S.-Zarembo '05)

(2) mode numbers



(3) fillings



Anomalous dimension

$$\gamma(g) = 2g^2 \sum_{k=1}^{K_4} \left(\frac{i}{x^+_{4,k}} - \frac{i}{x^-_{4,k}} \right)$$
$$= \sum_{l=1}^{\infty} \gamma_{2l}\, g^{2l}$$

- Many non-trivial checks up to 4 loops

(See, e.g. Beisert-Kristjansen-Staudacher '03, Beisert-Eden-Staudacher '06)

- Asymptotic Bethe ansatz

It breaks down when the wrapping interaction starts

(See, e.g. Kotikov-Lipatov-Rej-Staudacher-Velizhanin '07)

(Some part of finite size corrections can be systematically computed with the help of Lüscher formulas) (Janik-Łukowski '07)

Characteristic of the particle model:

$PSU(2|2) \times PSU(2|2) \ltimes \mathbb{R}^3$ centrally extended symmetry

Strings on $AdS_5 \times S^5$ in the uniform light-cone gauge

⇨ particle model with the same symmetry (Arutyunov, Frolov, Plefka, Zamaklar '05, '06)

- The symmetry fully determines
  - dispersion relation
  - S-matrix up to an overall scalar factor (Beisert '05)

- Determination of the remaining scalar factor

(Arutyunov-Frolov-Staudacher '04, Janik '06, Hernández-López '06, Beisert-Hernández-López '06, ...)

- Closed integral formula

(Beisert-Eden-Staudacher '06) (see also Dorey-Hofman-Maldacena '07)

Determination of the scalar factor

A) Factorized bootstrap program (phenomenological method)

crossing symmetry, poles and branch cuts,
perturbative computation, etc.

(Zamolodchikov^2 '77)

(Arutyunov-Frolov-Staudacher '04, Janik '06, Hernández-López '06, Beisert-Hernández-López '06, Beisert-Eden-Staudacher '06, ...)

B) Direct computation (microscopic derivation)

effective phase of underlying bare integrable model

(Korepin '79, Faddeev-Takhtajan '81, Andrei-Destri '84)

(KS-Satoh '07)

A) Factorized bootstrap program

Zamolodchikovs' derivation

i) Lie algebra and its representation

$$\hat{R}(u) = c_1(u)\hat{I} + c_2(u)\hat{P}$$

ii) unitarity, associativity (=Yang-Baxter Eqs.)

$$\hat{R}(u) = \frac{u}{u+i}\hat{I} + \frac{i}{u+i}\hat{P}$$

iii) crossing symmetry

$$\hat{S}(u) = X_{\mathrm{CDD}}(u)S_0(u)\hat{R}(u)$$

iv) pole analysis

$$\hat{S}(u) = S_0(u)\hat{R}(u)$$

AdS/CFT particle model

(i) String side

Mismatch between all-loop Bethe eqs. and classical strings

"three-loop discrepancy"

It can be repaired by a dressing factor in the Bethe eqs.

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{K} \sigma^2(u_j, u_k)\,\frac{u_k - u_j + i}{u_k - u_j - i}$$

AFS factor (Arutyunov-Frolov-Staudacher '04)

scalar factor of the S-matrix = dressing factor in the Bethe eqs.

AFS factor: correct at the leading semi-classical order

Quantum corrections in the worldsheet theory : $\frac{1}{\sqrt{\lambda}}$ expansion

(Hernández-López '06)

(Freyhult-Kristjansen '06)

(Gromov-Vieira '07)

All order conjecture

(Beisert-Hernández-López '06)

consistent with

(Arutyunov-Frolov '06)

Crossing symmetry

(Janik '06)

(ii) Gauge theory side

Low twist operators

$$\mathcal{O} = \mathrm{Tr}(D^S Z^L) + \cdots \qquad\qquad S \gg L(= 2, 3, \ldots)$$

soft(cusp) anomalous dimension:

$$\Delta = S + f(g) \log S + \mathcal{O}(S^0)$$

$f(g)$: universal scaling function

⇅ (Eden-Staudacher '06)
(Beisert-Eden-Staudacher '06)

$S_0(p_1, p_2; g)$: scalar factor

trivial up to three loops

Proposal of Beisert-Eden-Staudacher

1/2 of the expected phase

- Based on phenomenological principles:

$$\left\{\begin{array}{l}\text{scaling law (Beisert-Klose '05)}\\ \text{transcendentality (Kotikov-Lipatov '02)}\\ \text{cancellation of } \zeta(2n+1)\end{array}\right\} \Longrightarrow$$

A phase factor is uniquely fixed order by order

- "Analytic continuation" from the string side
- Numerical tests against MHV amplitudes at 4 loops

(Bern-Czakon-Dixon-Kosower-Smirnov '06) (Cachazo-Spradlin-Volovich '06)

Closed integral formula (in the Fourier space)

$$\hat{K}_d(t,t') = 8g^2 \int_0^\infty dt'' \hat{K}_1(t, 2gt'') \frac{t''}{e^{t''}-1} \hat{K}_0(2gt'', t)$$

$$\hat{K}_0(t,t') = \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2} \qquad \hat{K}_1(t,t') = \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2}$$

- How to understand its structural simplicity?
- Any simple derivation/interpretation?

Emergence of such integral kernels
in solving the nested levels of Bethe eqs.

(Rej-Staudacher-Zieme '07)

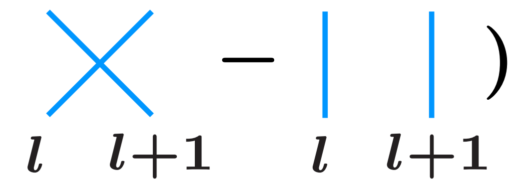
B) Direct computation

su(2) R-matrix

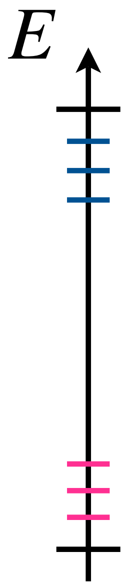
⇨ BAE for Heisenberg spin-chain

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L=\prod_{l\neq k}^{J}\frac{u_k-u_l+i}{u_k-u_l-i}$$

$$H=\sum_{l=1}^{L}($$



$)$ anti-ferromagnetic chain



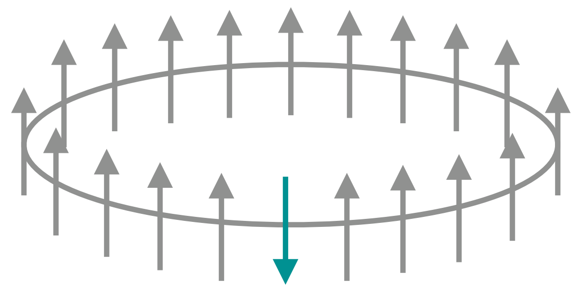
ferromagnetic state

magnon states

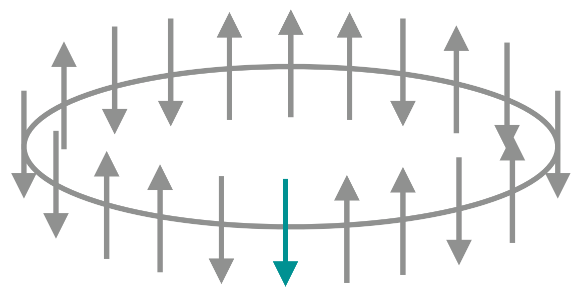
spinon states ⇨ fundamental excitations

anti-ferromagnetic state ⇨ vacuum

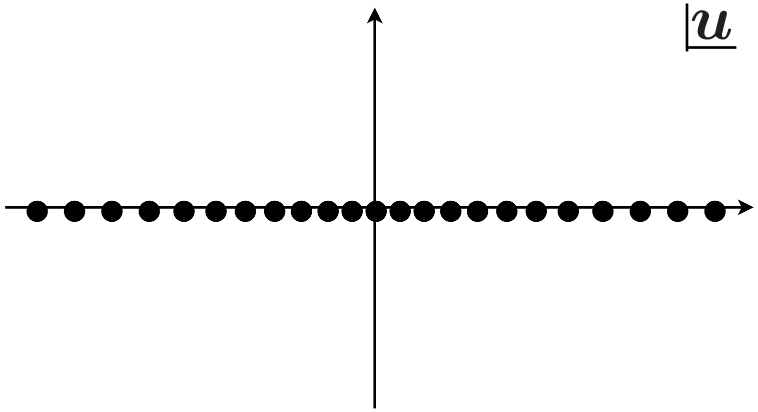
Single magnon state

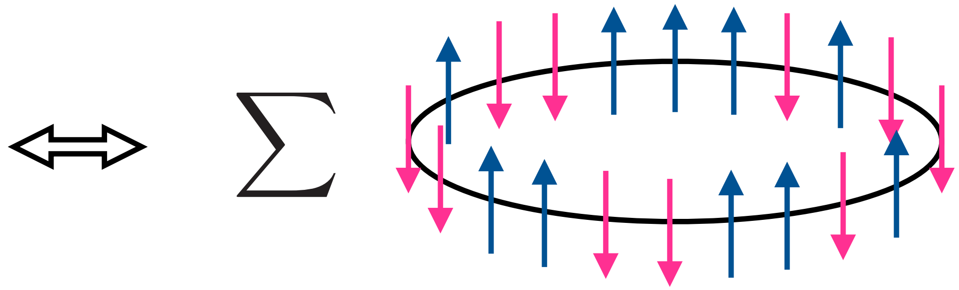


Single spinon state

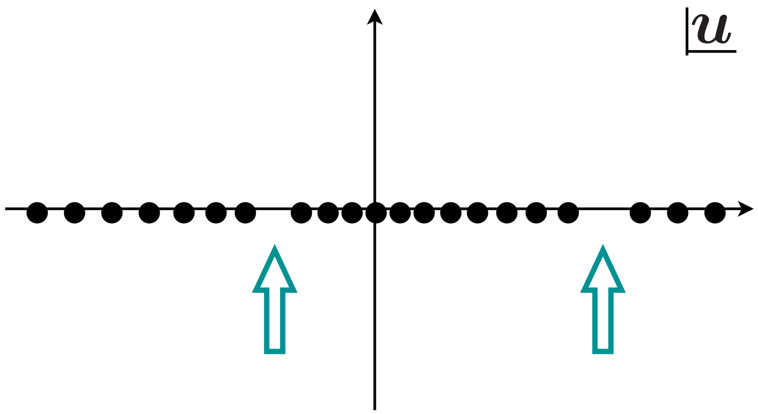


antiferromagnetic ground states





2-spinon excitated states: 2-holes



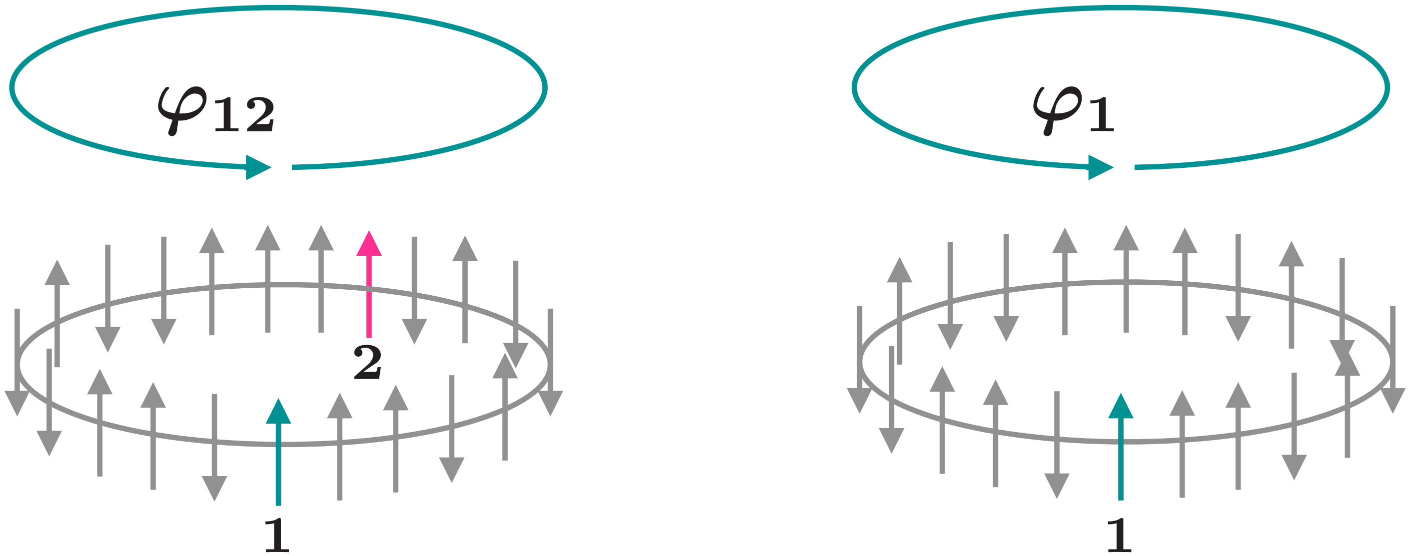
scattering phase of the 2-spinons

$$\Rightarrow \quad S_0(u) = i\frac{\Gamma(-\frac{u}{2i})\Gamma(\frac{1}{2}+\frac{u}{2i})}{\Gamma(\frac{u}{2i})\Gamma(\frac{1}{2}-\frac{u}{2i})}$$

Scalar factor of the Zamolodchikovs' S-matrix

How to compute the scattering phase?

Total scattering phase that the particle 1 acquires in the presence/absence of the particle 2



$$\varphi_{12} - \varphi_1 = \delta_{\text{bare}} + \delta_{\text{back-reaction}}$$

ln (R-matrix) $\ln S_0$

- $\mathfrak{su}(2)$ R-matrix

  ⇒ Bethe equations for the $\mathfrak{su}(2)$ Heisenberg spin-chain

antiferromagnetic vacuum $\sim \uparrow\downarrow$

  ⇒ Zamolodchikovs' S-matrix

- $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ R-matrix

  ⇒ Asymptotic all-loop $\mathfrak{psu}(2,2|4)$ Bethe equations (without the dressing phase)

  (Beisert-Staudacher '05)

"antiferromagnetic" vacuum $\sim |\phi_1\phi_2 Z^+\rangle + |\psi_1\psi_2\rangle$

  ⇒ $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ S-matrix with the dressing factor

  (KS-Satoh '07) (cf. Rej-Staudacher-Zieme '07)

All-order Bethe equations (Beisert-Staudacher '05)

$$1 = \prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-} \,,$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + i/2}{u_{1,k} - u_{2,j} - i/2} \prod_{j=1}^{K_4} \frac{1 - g^2/x_{1,k}\, x_{4,j}^+}{1 - g^2/x_{1,k}\, x_{4,j}^-} \,,$$

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + i/2}{u_{2,k} - u_{3,j} - i/2} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + i/2}{u_{2,k} - u_{1,j} - i/2} \,,$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + i/2}{u_{3,k} - u_{2,j} - i/2} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-} \,,$$

$$\left(\frac{x_{4,k}^+}{x_{4,k}^-}\right)^J = \prod_{j\neq k}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i}\, \underbrace{e^{2i\theta(u_{4,k},u_{4,j})}}_{=\,1} \prod_{j=1}^{K_1} \frac{1 - g^2/x_{4,k}^-\, x_{1,j}}{1 - g^2/x_{4,k}^+\, x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}}$$

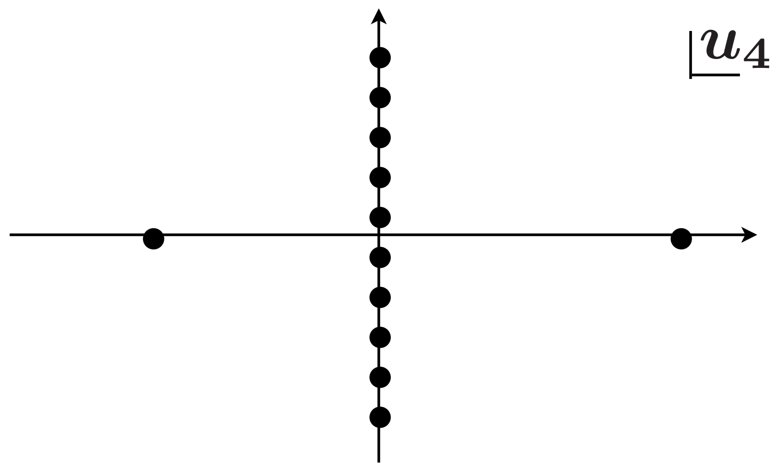
$$\times \prod_{j=1}^{K_7} \frac{1 - g^2/x_{4,k}^-\, x_{7,j}}{1 - g^2/x_{4,k}^+\, x_{7,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}} \,,$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + i/2}{u_{5,k} - u_{6,j} - i/2} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^+}{x_{5,k} - x_{4,j}^-} \,,$$

$$1 = \prod_{j\neq k}^{K_6} \frac{u_{6,k} - u_{6,j} - i}{u_{6,k} - u_{6,j} + i} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + i/2}{u_{6,k} - u_{5,j} - i/2} \prod_{j=1}^{K_7} \frac{u_{6,k} - u_{7,j} + i/2}{u_{6,k} - u_{7,j} - i/2} \,,$$

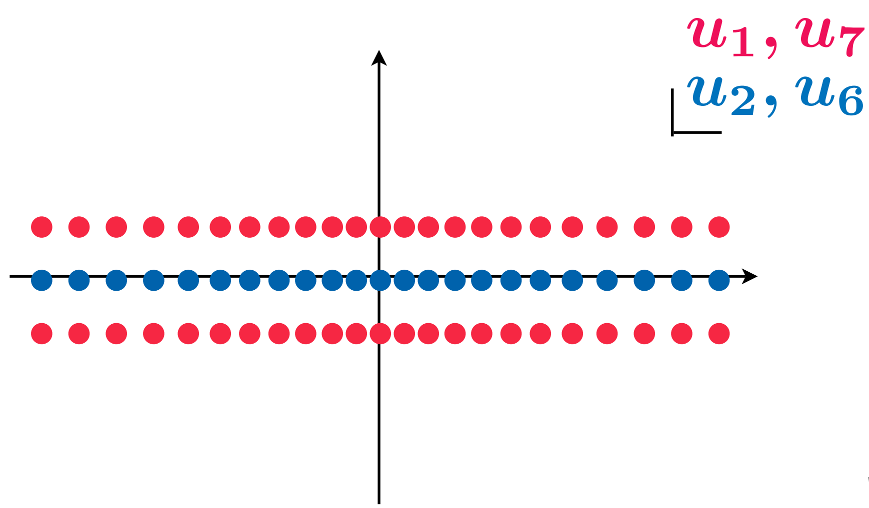
$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + i/2}{u_{7,k} - u_{6,j} - i/2} \prod_{j=1}^{K_4} \frac{1 - g^2/x_{7,k}\, x_{4,j}^+}{1 - g^2/x_{7,k}\, x_{4,j}^-} \,.$$

2 fundamental excitations over the physical vacuum



$$L \longrightarrow \infty$$
$$M \longrightarrow \infty$$

$2M$ u_4's + 2 excitation u_4's $\Longleftrightarrow$ (16 dim irrep.)^2



M stacks

$$\hat{S} = S_0^2 [\hat{R}_{\mathfrak{su}(2|2)} \otimes \hat{R}_{\mathfrak{su}(2|2)}]$$

S-matrix with the dressing phase

- We proposed a possible form of the microscopic derivation of the S-matrix in AdS/CFT
- S-matrix, including the overall scalar factor, is completely determined by the $\mathfrak{su}(2|2)$ symmetry

No need of gauge/string perturbative data

- Once the integrability is proven both in the Planar $\mathcal{N} = 4$ super Yang-Mills and in the free superstrings on AdS, the spectrum is uniquely constructed for arbitrary λ.

$$\Downarrow$$

Quantitative "proof" of the AdS/CFT correspondence in the limit $N \to \infty, L \to \infty$

Summary

- The spectral problem of the dilatation operator is fully solved at one-loop
- General solutions of classical strings on the AdS background can be constructed
- Spectra of all-order dilatation operator / quantum strings are partly available

Prospects

- Wrapping interactions and finite size corrections
  - Lüscher formulas (Janik-Łukowski '07)
  - Thermodynamic Bethe ansatz (Arutyunov-Frolov '07)
- Proof/disproof of integrability
  - Yangian symmetry (Beisert-Erkal '07)
  - Non planar case? (Casteill-Janik-Jarosz-Kristjansen '07)

Appendix

Starting point: $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ R-matrix

(S-matrix without the dressing factor)

$$\hat{S} = S_0^2 [\hat{R}_{\mathfrak{su}(2|2)} \otimes \hat{R}_{\mathfrak{su}(2|2)}]$$

$$S_0(p_k, p_j)^2 = \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \, e^{2i\theta(u_k, u_j)}$$

$e^{2i\theta(u_k,u_j)} = 1$

- Periodic boundary condition

Yang equations: $$e^{ip_k L} = \prod_{j \neq k}^{K_4} \hat{S}(p_k, p_j)$$

$\Downarrow$ diagonalization (by nested Bethe ansatz) (Beisert '05, Martins-Melo '07, de Leeuw '07, ...)

Asymptotic all-order $\mathfrak{psu}(2,2|4)$ Bethe equations

(Here: no direct correspondence with Yang-Mills operators)

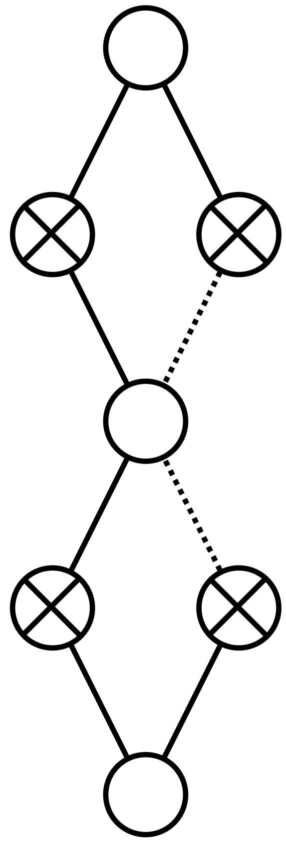
Rapidity variables

$$x^{\pm}(u) = x(u \pm \tfrac{i}{2})$$

$$x(u) = \frac{u}{2}\left(1 + \sqrt{1 - 4g^2/u^2}\right)$$

$$g = \frac{\sqrt{\lambda}}{4\pi}$$

Constraints on the occupation numbers



K_2

$\leq$

$K_1 + K_3$

$\leq$

K_4

$\geq$

$K_5 + K_7$

$\geq$

K_6

How to construct the "anti-ferromagnetic" vacuum?

$$\prod_{j=1}^{K_4} \frac{1 - g^2/x_{7,l}\, x^+_{4,j}}{1 - g^2/x_{7,l}\, x^-_{4,j}} = \prod_{j=1}^{K_6} \frac{u_{7,l} - u_{6,j} - i/2}{u_{7,l} - u_{6,j} + i/2}$$

$$\prod_{j=1}^{K_7} \frac{u_{6,l} - u_{7,j} + i/2}{u_{6,l} - u_{7,j} - i/2} = \prod_{j\neq l}^{K_6} \frac{u_{6,l} - u_{6,j} + i}{u_{6,l} - u_{6,j} - i}$$

$$\wr\wr$$

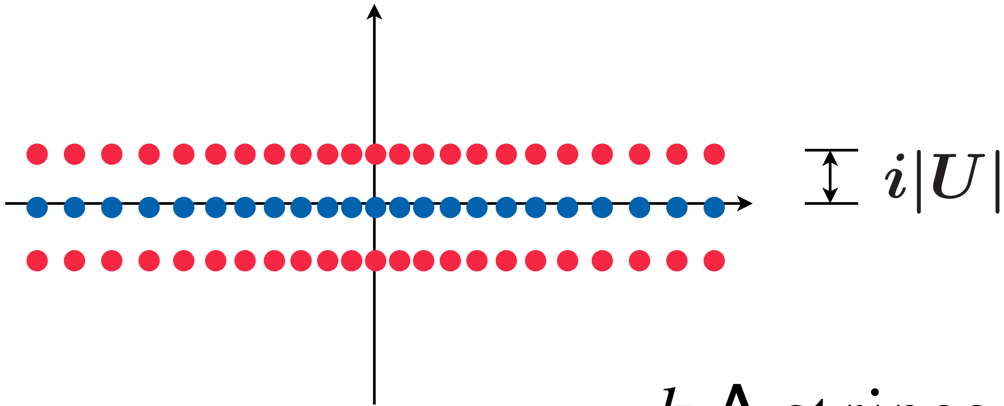
$$e^{ik_l L_H} = \prod_{j=1}^{M} \frac{\sin k_l - \Lambda_j - i|U|}{\sin k_l - \Lambda_j + i|U|}$$

$$\prod_{j=1}^{N_e} \frac{\Lambda_l - \sin k_j + i|U|}{\Lambda_l - \sin k_j - i|U|} = \prod_{j\neq l}^{M} \frac{\Lambda_l - \Lambda_j + 2i|U|}{\Lambda_l - \Lambda_j - 2i|U|}$$

Lieb-Wu equations for the Hubbard model in the attractive case $(U < 0)$

Ground state configuration

(Woynarovich ’83, Essler-Korepin ’94)



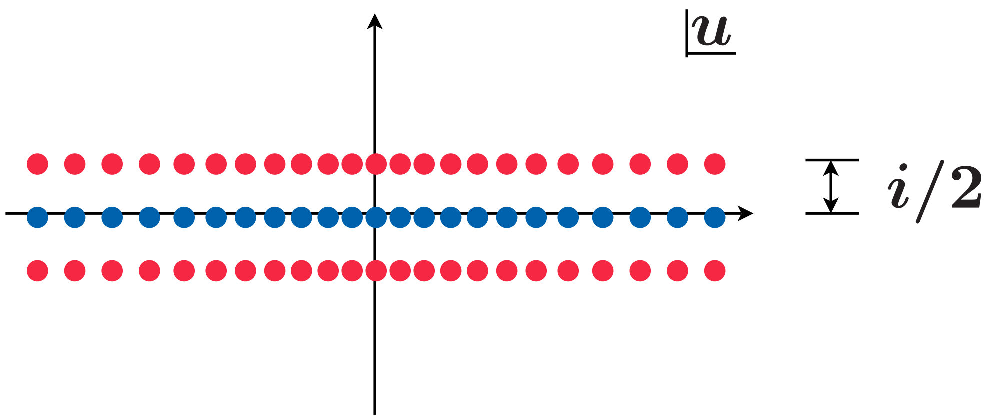
• Λ

• $\sin k$

Hubbard model
(attractive case)

k-Λ strings

⇓



• u_6

• u_7

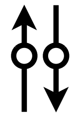
‘AF vacuum’
of the bare model

stacks

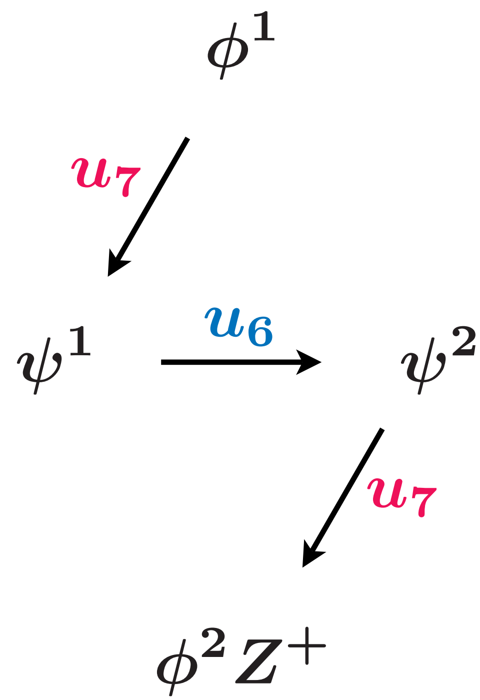
(Rej-Staudacher-Zieme ’07) (KS-Satoh ’07)

(Beisert-Kazakov-KS-Zarembo ’05)

- k-Λ string
  - $\sin k$
  - Λ
  - $\sin k$

$\Longleftrightarrow$  bound state of electrons

- stack
  - u_7
  - u_6
  - u_7



$$|\cdots \phi^1 \phi^1 \cdots\rangle \to |\cdots \phi^1 \phi^2 Z^+ \cdots\rangle + |\cdots \psi^1 \psi^2 \cdots\rangle$$

Correspondence of occupation numbers

$K_4 \Leftrightarrow L_H$ (length of the Hubbard model)

$K_6 \Leftrightarrow M$ (# of down spins $\downarrow$)

$K_7 \Leftrightarrow N_e$ (# of electrons $\uparrow$ & $\downarrow$)

Ground state of the Hubbard model

$N_e = L_H$ charge-singlet (half-filled)

$M = N_e/2$ spin-singlet $\uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow$

$\Rightarrow$

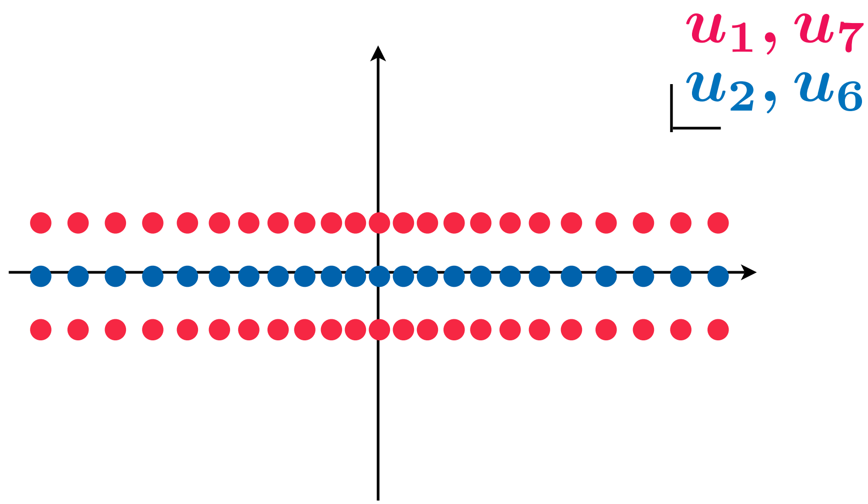
Occupation numbers for the 'AF vacuum'

$$(K_1, K_2, K_3, K_4, K_5, K_6, K_7) = (2M, M, 0, 2M, 0, M, 2M)$$

Stack configuration
at nested levels

⇕

Insertion of
the dressing phase

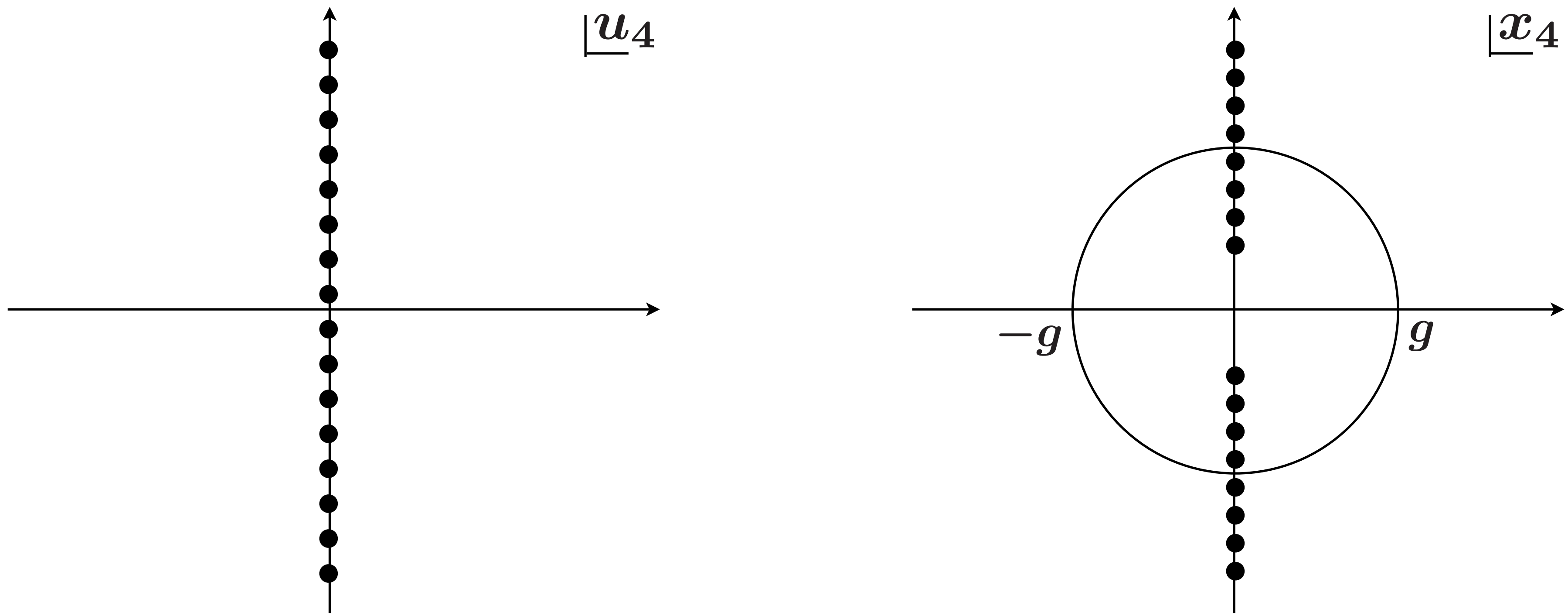


M stacks

(KS-Satoh '07) (Rej-Staudacher-Zieme '07)

- In order to support the stack structure, one needs additional $2M$ u_4 roots.

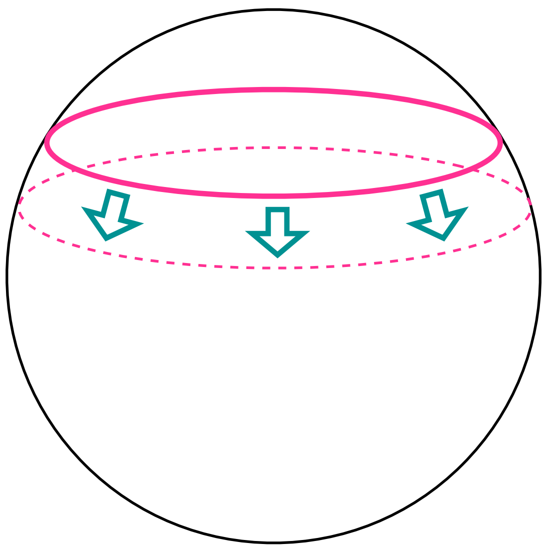
Configuration of the central roots



$$\left(u \pm \frac{i}{2} = x^{\pm} + \frac{g^2}{x^{\pm}} \right)$$

This configuration, when considered in the physical Bethe equations, corresponds to the pulsating string in S^2

pulsating string



point-like string