Deconstruction, lattice SUSY and noncommutative geometry

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at RIKEN topical workshop,
"Lattice chiral fermions and geometry in matrix models", Apr. 26, 2003

— Reference —

• J.N., S.-J. Rey and F. Sugino,
JHEP **02**, 032 (2003) [hep-th/0301025].

0. Introduction

- lattice chirality
  - introduce extra dimension (Kaplan '92)
  - realize chiral fermions on the boundaries

- lattice SUSY (Kaplan-Katz-Ünsal '02)
  based on idea of "deconstruction"
  (Arkani-Hamed–Cohen–Georgi '01)
  - create a new dimension (latticized) from internal degrees of freedom
  - original motivation : a UV completion of a 5d theory
  - finite N matrix models can accomodate exact SUSY
    $\Longrightarrow$ SYM in (1+1)d, (2+1)d, (3+1)d without fine-tuning

- new interplays between lattice and matrix
  - twisted reduced models
    $\Longleftrightarrow$ lattice field theories
    on noncommutative geometry (NCG)
    (Aoki-Ishibashi-Iso-Kawai-Kitazawa-Tada '99,
    Bars-Minic '99, Ambjørn-Makeenko-J.N.-Szabo '99, '00)
  - Monte Carlo sim. of matrix models
    (Bietenholz-Hofheinz-J.N. '02, Ambjørn-Catterall '02)
  - Ginsparg-Wilson fermions on NCG
    (J.N.–Vazquez-Mozo '01, Aoki-Iso-Nagao '02)
  - topological issues on NCG
    (Iso-Nagao '02)
  - quenched reduced models
    as large N gauge theory
    (Kiskis-Narayanan-Neuberger '02, Kikukawa-Suzuki '02)
- further studies may benefit both sides !

Plan of the talk

0. Introduction

1. deconstruction

2. realization in matrix model (“orbifolding”)

   toroidal compactification of M-theory

3. SYM on the lattice : $(1+1)$d example

4. generalization to higher dimensions

5. noncommutative geometry

6. Summary and Discussions

1. deconstruction (Arkani-Hamed–Cohen–Georgi '01)

4d theory with SU$(k)^N$ gauge group
with bifundamental complex scalar field

$$\Phi_n \rightarrow g_n \Phi_n g_{n+1}^\dagger \quad (1)$$

$$D_\mu \Phi_n = \partial_\mu \Phi_n - i A_\mu^n \Phi_n + i \Phi_n A_\mu^{n+1} \quad (2)$$

$$S = \int d^4x \left[-\frac{1}{2g^2} \sum_{n=1}^{N} \mathrm{tr} F_{n\mu\nu}^2 + \sum_{n=1}^{N} \mathrm{tr}(D_\mu \Phi_n)^\dagger D_\mu \Phi_n + \sum_{n=1}^{N} \mathrm{tr}(\Phi_n^\dagger \Phi_n - f^2 \mathbf{1})^2 \right]$$

"moose" (or "quiver") diagram

At low energy :

$$\langle \Phi_n(x) \rangle = f U_n \quad ; \quad U_n \in \mathrm{U}(k) \quad (3)$$

$\longrightarrow$ 5d SU(k) gauge theory
with latticized 5-th dimension

- VEV of Φ_n
  $\Longrightarrow$ "hopping" in extra dimensions

- From the viewpoint of 4d theory

  - Higgs mechanism
    $$\underbrace{\mathrm{SU}(k) \times \cdots \times \mathrm{SU}(k)}_{N} \longrightarrow \mathrm{SU}(k)$$

  - massive gauge bosons = KK excitations

- Lorentz inv. restored in the cont. lim.
  by taking lattice spacing : $a = \frac{1}{f}$

- locality in 5-th direction
  $\longleftarrow$ particular choice of interaction
  represented by the moose diagram

2. realization in matrix models

2.1 rough idea

clock and shift matrices

$$\omega = \exp(2\pi i/N)$$

$$Q = \begin{pmatrix} \mathbf{1}_k & & & \\ & \omega \mathbf{1}_k & & \\ & & \ddots & \\ & & & \omega^{N-1}\mathbf{1}_k \end{pmatrix} \qquad (4)$$

$$P = \begin{pmatrix} 0 & \mathbf{1}_k & & & 0 \\ & 0 & \mathbf{1}_k & & \\ & & \ddots & \ddots & \\ & & & \ddots & \mathbf{1}_k \\ \mathbf{1}_k & & & & 0 \end{pmatrix} \qquad (5)$$

satisfying 't Hooft-Weyl algebra

$$QP = \omega PQ \qquad (6)$$

"orbifolding" : $Q^\dagger \Phi Q = \Phi$
$\Longrightarrow$ Φ becomes block-diagonal

$$\Phi = \begin{pmatrix} \tilde{\Phi}_1 & & & \\ & \tilde{\Phi}_2 & & \\ & & \ddots & \\ & & & \tilde{\Phi}_N \end{pmatrix} \qquad (7)$$

P shifts the diagonal blocks

$$P^\dagger \Phi P = \begin{pmatrix} \tilde{\Phi}_N & & & \\ & \tilde{\Phi}_1 & & \\ & & \ddots & \\ & & & \tilde{\Phi}_{N-1} \end{pmatrix} \qquad (8)$$

moose diagram can be realized as

$$\mathrm{Tr}[\Phi(P^\dagger \Phi P)] = \sum_{n=1}^{N} \mathrm{tr}(\tilde{\Phi}_n \tilde{\Phi}_{n+1}) \qquad (9)$$

$$\tilde{\Phi}_{N+1} = \tilde{\Phi}_1 \text{ (periodic)}$$

space-time indices
gauge indices } treated on equal footing

relation to NCG (see later)

2.2 toroidal compactification of M-theory

(Taylor '96)

IKKT model (bosonic part)

$$S = -\mathrm{Tr}\left([X_\mu, X_\nu]^2\right) \tag{10}$$

X_μ ($\mu = 1, \cdots, 10$) : hermitian matrices

toroidal compactification in 1-direction

$$\begin{cases} \Omega X_1 \Omega^\dagger &= X_1 + R\,\mathbf{1} \\ \Omega X_j \Omega^\dagger &= X_j \qquad (j \geq 2) \end{cases} \tag{11}$$

$$(\Omega f)(s) = \mathrm{e}^{is} f(s) \qquad (0 \leq s < 2\pi) \tag{12}$$

$$\text{solution} : \begin{cases} X_1 &= i\,R\,\frac{\partial}{\partial s} + A(s) \\ X_j &= Y_j(s) \qquad (j \geq 2) \end{cases} \tag{13}$$

$$S = \int ds\, \mathrm{tr}\left\{R^2 (\nabla Y_i(s))^2 - [Y_i, Y_j]^2\right\} \tag{14}$$

$$\nabla = \frac{\partial}{\partial s} - \frac{i}{R} A(s) \tag{15}$$

10d YM reduced to 1 dim.

consistent with T-duality

2.3 finite N version

Eguchi-Kawai model

$$S = -\mathrm{Tr}(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger) \qquad (16)$$

$U_\mu = \mathrm{e}^{iX_\mu} \Longrightarrow$ Hermitian model

"toroidal compactification" in 1-direction

$$\begin{cases} QU_1Q^\dagger = \omega U_1 \\ QU_jQ^\dagger = U_j \qquad (j \geq 2) \end{cases} \qquad (17)$$

$U_1 = P$ (particular solution)
$U_1 = U_1' P \qquad \Longrightarrow \qquad QU_1'Q^\dagger = U_1'$

general solution

$$U_1' = \begin{pmatrix} \tilde{U}_1^{(1)} & & & \\ & \tilde{U}_1^{(2)} & & \\ & & \ddots & \\ & & & \tilde{U}_1^{(N)} \end{pmatrix} \qquad (18)$$

$$U_j = \begin{pmatrix} \tilde{U}_j^{(1)} & & & \\ & \tilde{U}_j^{(2)} & & \\ & & \ddots & \\ & & & \tilde{U}_j^{(N)} \end{pmatrix} \qquad (19)$$

The action becomes

$$S = -\sum_{n=1}^{N}\sum_{j=2}^{10} \mathrm{tr}(\tilde{U}_1^{(n)}\tilde{U}_j^{(n+1)}\tilde{U}_1^{(n)\dagger}\tilde{U}_j^{(n)\dagger}) - \sum_{n=1}^{N}\sum_{i,j\geq 2}^{10} \mathrm{tr}(\tilde{U}_i^{(n)}\tilde{U}_j^{(n)}\tilde{U}_i^{(n)\dagger}\tilde{U}_j^{(n)\dagger})$$

gauge symmetry $\subset$ U(kN)

$$\begin{cases} \tilde{U}_1^{(n)} \rightarrow g^{(n)}\tilde{U}_1^{(n)}g^{(n+1)\dagger} & : \text{link} \\ \tilde{U}_j^{(n)} \rightarrow g^{(n)}\tilde{U}_j^{(n)}g^{(n)\dagger} \quad (j \geq 2) & : \text{site} \end{cases} \tag{20}$$

"Deconstruction" is realized naturally.

But unitary matrices cannot accomodate SUSY.

Can finite N Hermitian matrix model
leads to any sensible lattice theory
via orbifolding ?

note : no. of d.o.f. is the same...

3. SYM on the lattice : $(1+1)$d

(Kaplan-Katz-Ünsal '02)

target theory :

$(3+1)$d $\mathcal{N}=1$ SYM reduced to $(1+1)$d

$$\begin{aligned}\mathcal{L} &= \mathrm{Tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \overline{\Psi} i \not{D} \Psi \\ &\quad -(D_\mu S)^\dagger (D^\mu S) + \sqrt{2}(\overline{\Psi}_L [S, \Psi_R] + h.c.) \\ &\quad -\frac{1}{2}[S^\dagger, S]^2\Big)\end{aligned}$$

(v_0, v_1) : gauge field

$$\left.\begin{array}{lcl} S & : & \text{complex scalar} \\ \Psi & : & \text{2-comp. Dirac} \end{array}\right\} \quad \text{adjoint rep.}$$

Mother theory :

$(3+1)$d $\mathcal{N}=1$ SYM reduced to $(0+1)$d

superfield notation in $(1+1)$d $\longrightarrow$ $(0+1)$d

$$\begin{aligned} V &= (v_0 - \sigma) - 2i\theta\bar{\lambda} - 2i\bar{\theta}\lambda - 2\bar{\theta}\theta d \\ \Phi &= \phi + \sqrt{2}\,\theta\psi + i\bar{\theta}\theta\dot{\phi} \end{aligned}$$

Orbifolding :

$$Q^\dagger V Q = V \tag{21}$$
$$Q^\dagger \Phi Q = \omega \Phi \tag{22}$$

Daughter theory :

$$\begin{aligned} L &= \sum_n \mathrm{tr}\Big[\frac{1}{2}(D_0\sigma_n)^2 + \bar\lambda_n\, iD_0\,\lambda_n \\ &\quad +|D_0\,\phi_n|^2 + \bar\psi_n\, iD_0\psi_n \\ &\quad -\bar\lambda_n[\sigma_n,\lambda_n] + \bar\psi_n(\sigma_n\psi_n - \psi_n\sigma_{n+1}) \\ &\quad -\sqrt{2}\left(i\bar\phi_n(\lambda_n\psi_n + \psi_n\lambda_{n+1}) + h.c.\right) \\ &\quad -\left|\sigma_n\phi_n - \phi_n\sigma_{n+1}\right|^2 \\ &\quad -\frac{1}{2}\left(\phi_n\bar\phi_n - \bar\phi_{n+1}\phi_{n+1}\right)^2\Big] \\ D_0\sigma_n &= \partial_0\sigma_n + iv_{0,n}\sigma_n - i\sigma_n v_{0,n} \\ D_0\phi_n &= \partial_0\phi_n + iv_{0,n}\phi_n - i\phi_n v_{0,n+1} \quad \text{etc.} \end{aligned}$$

classical moduli space :

$$\phi_n(t) = \mathrm{diag}(\lambda_n^{(1)}, \cdots, \lambda_n^{(k)}) \tag{23}$$

expand around the U(k) symmetric point :

$$\phi_n(t) = \frac{f}{\sqrt{2}}\mathbf{1}_k \tag{24}$$

lattice spacing : $a = 1/f$

decompose ϕ as

$$\phi = \frac{h^{(1)} + ih^{(2)}}{\sqrt{2}} \tag{25}$$

and identify

$$v_1 = h^{(2)}\ , \quad S = \frac{\sigma + ih^{(1)}}{\sqrt{2}}\ , \quad \Psi = \begin{pmatrix} \psi \\ -i\bar{\lambda} \end{pmatrix} \tag{26}$$

$\Longrightarrow$ target theory (classical continuum limit)

Problems:

classical moduli space in $(0+1)$d SYM : unstable quantum mechanically

But "radion" has mass of $O(N)$
$\Longrightarrow$ fix initial/final conditions properly
Does it really work ?

recovery of full SUSY
$\longleftarrow$ perturbative power counting argument
Nonperturbatively OK ?

4. generalization to higher dim.

Consider : $S = \mathrm{Tr}(\Phi_1 \cdots \Phi_M)$

Orbifolding :

$$\mathcal{Q}_a \Phi_j \mathcal{Q}_a^\dagger = \omega^{r_{j,a}} \Phi_j \quad ; \quad a = 1, 2, \cdots, d \qquad (27)$$

matrix size $= N^d \cdot k$

$$\begin{array}{ccccccccccccc}
\mathcal{Q}_1 & = & Q & \otimes & \mathbf{1}_N & \otimes & \cdots & \otimes & \mathbf{1}_N & \otimes & \mathbf{1}_k \\
\mathcal{Q}_2 & = & \mathbf{1}_N & \otimes & Q & \otimes & \cdots & \otimes & \mathbf{1}_N & \otimes & \mathbf{1}_k \\
 & \vdots & & & & & & & & & \\
\mathcal{Q}_d & = & \mathbf{1}_N & \otimes & \mathbf{1}_N & \otimes & \cdots & \otimes & Q & \otimes & \mathbf{1}_k
\end{array}$$

$$\mathcal{P}(\vec{n}) = P^{n_1} \otimes P^{n_2} \otimes \cdots \otimes P^{n_d} \otimes \mathbf{1}_k$$

particular solution : $\Phi_j = \mathcal{P}(\vec{r}_j)$

$\Phi_j = \Phi'_j \mathcal{P}(\vec{r}_j) \quad \Longrightarrow \quad \mathcal{Q}_a \Phi'_j \mathcal{Q}_a^\dagger = \Phi'_j$

general solution :

$$\Phi'_j = \sum_{\vec{n}} \Delta(\vec{n}) \otimes \phi_j(\vec{n}) \qquad (28)$$

$\Delta(\vec{n}) = J^{(n_1)} \otimes \cdots \otimes J^{(n_d)}$

$(J^{(n)})_{ij} = \delta_{in}\delta_{jn}$

$$\mathcal{P}(\vec{m})\Delta(\vec{n})\mathcal{P}(\vec{m})^\dagger = \Delta(\vec{n}+\vec{m})$$

The action becomes

$$\begin{aligned} S &= \sum_{\vec{n}} \mathrm{tr}[\phi_1(\vec{n})\phi_2(\vec{n}+\vec{r}_1)\cdots \\ &\qquad \cdots\phi_M(\vec{n}+\vec{r}_1+\cdots+\vec{r}_{M-1})] \end{aligned}$$

gauge symmetry $\subset \mathrm{U}(N^d \cdot k)$

$$\phi_i(\vec{n}) \quad \longmapsto \quad g(\vec{n})\,\phi_i(\vec{n})\,g^\dagger(\vec{n}+\vec{r}_i) \qquad (29)$$

$\phi_i(\vec{n})$: link connecting $\vec{n}$ and $\vec{n}+\vec{r}_i$
(site, if $\vec{r}_i = 0$)

$\sum_j \vec{r}_j = 0$ is necessary (otherwize $S \equiv 0$)

A convenient choice for $\vec{r}_j$:
lin. comb. of charges associated with R sym.

rank of R sym. of the reduced model
= maximum lattice dimension

no. of fermions with $\vec{r} = 0$
= no. of unbroken supercharges

5. Noncommutative geometry

5.1 a different type of compactification

(Connes-Douglas-Schwarz '97)

IKKT model : $S = -\mathrm{Tr}\,[X_\mu, X_\nu]^2$

$$\Omega_a X_\mu \Omega_a^\dagger = X_\mu + 2\pi\delta_{\mu a} \qquad (30)$$

$$\begin{aligned} \Omega_1 &= \mathrm{e}^{i\gamma s} \otimes (\tilde{\Gamma}_1)^{\dagger p} \\ \Omega_2 &= \mathrm{e}^{2\pi i \partial_s} \otimes (\tilde{\Gamma}_2)^\dagger \end{aligned} \qquad (31)$$

$\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$: $q \times q$ clock & shift matrices

General solution : $X_\mu = X_\mu^{(0)} + A_\mu$

$$\begin{aligned} X_1^{(0)} &= 2\pi i \frac{1}{\gamma} \partial_s \otimes \mathbf{1}_q \\ X_2^{(0)} &= s \otimes \mathbf{1}_q \\ A_\mu &= \int \mathrm{d}x A_\mu(x) \Delta(x) \end{aligned} \qquad (32)$$

$\Delta(x)$: complete basis

$$\Delta(x) = \sum_{\vec{k}} (Z_1)^{k_1} (Z_2)^{k_2} \mathrm{e}^{-i\theta\epsilon_{\mu\nu}k_\mu k_\nu/2} \mathrm{e}^{i\vec{k}\cdot\vec{x}}$$

$$Z_1 = \mathrm{e}^{i\frac{1}{q}s} \otimes (\tilde{\Gamma}_1)^\dagger$$

$$Z_2 = \mathrm{e}^{-\frac{2\pi i}{\gamma q}\partial_s} \otimes (\tilde{\Gamma}_2)^a, \qquad (33)$$

where $ap - bq = 1$ $(\exists b)$.

$$Z_\mu \Omega_\nu = \Omega_\nu Z_\mu \qquad (34)$$

$$Z_1 Z_2 = \mathrm{e}^{-i\theta} Z_2 Z_1 \qquad (35)$$

identifying $Z_\mu = \mathrm{e}^{-i\hat{x}_\mu}$,

$$[\hat{x}_1, \hat{x}_2] = i\theta \qquad \Longrightarrow \text{NCG !} \qquad (36)$$

$X_a^{(0)}$: derivative op. on 2d NC torus; i.e.,

$$[X_a^{(0)}, \Delta(x)] = i\frac{2\pi}{\gamma q}\frac{\partial}{\partial x_a}\Delta(x) \qquad (37)$$

IKKT model $\Longrightarrow$ 2d NCYM

5.2 finite dimensional version

(Ambjørn-Makeenko-J.N.-Szabo '99)

Take $\gamma = \frac{m}{nq}$

matrix size : $N = (mnq) \times q$

$$
\begin{aligned}
\Omega_1 &= (\Gamma_2)^m \otimes (\tilde{\Gamma}_1)^{\dagger p} \\
\Omega_2 &= (\Gamma_1)^m \otimes (\tilde{\Gamma}_2)^{\dagger} \\
Z_1 &= (\Gamma_2)^n \otimes (\tilde{\Gamma}_1)^{\dagger} \\
Z_2 &= (\Gamma_1)^{\dagger n} \otimes (\tilde{\Gamma}_2)^{a}
\end{aligned} \tag{38}
$$

Action and orbifolding condition :

$$
\begin{aligned}
S &= \sum_{\mu \neq \nu} Z_{\mu\nu} \mathrm{Tr}\left[U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right] \\
\Omega_\nu U_\mu \Omega_\nu^\dagger &= \mathrm{e}^{2\pi i \delta_{\mu\nu}/(nq)} U_\mu
\end{aligned} \tag{39}
$$

particular solution :

$$
\begin{aligned}
U_1^{(0)} &= (\Gamma_1)^\dagger \otimes \mathbf{1}_q \\
U_2^{(0)} &= \Gamma_2 \otimes \mathbf{1}_q
\end{aligned} \tag{40}
$$

General solution : $U_\mu = U'_\mu U_\mu^{(0)}$

$$U'_\mu = \sum_x U_\mu(x)\Delta(x)$$
$$\Delta(x) = \sum_{\vec{k}} (Z_1)^{k_1}(Z_2)^{k_2} \mathrm{e}^{-i\theta\epsilon_{\mu\nu}k_\mu k_\nu/2}\mathrm{e}^{i\vec{k}\cdot\vec{x}}$$

$U_\mu^{(0)}$: lattice shift operator; i.e.,

$$U_\mu^{(0)}\Delta(x)U_\mu^{(0)\dagger} = \Delta(x-\widehat{\mu}) \qquad (41)$$

$$S = \sum_{\mu\neq\nu} Z_{\mu\nu} \mathrm{Tr}\left[U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right]$$
$$= \sum_{\mu\neq\nu} \mathrm{Tr}\left[U'_\mu (U_\mu^{(0)} U'_\nu U_\mu^{(0)\dagger})(U_\nu^{(0)} U_\mu'^\dagger U_\nu^{(0)\dagger}) U_\nu'^\dagger\right]$$
$$= \sum_x \mathrm{tr}\left(U_\mu(x) \star U_\nu(x+\epsilon\widehat{\mu}) \star U_\mu(x+\epsilon\widehat{\nu})^\dagger \star U_\nu(x)^\dagger\right)$$

twisted Eguchi-Kawai model $\Longrightarrow$
NC version of Wilson's lattice gauge theory

Using the same orbifolding condition,
IKKT model $\Longrightarrow$ SYM on the NC lattice

(J.N., S.-J. Rey and F. Sugino '03)

5.3 twisted reduced models and NCG

IKKT model : $S = -\mathrm{Tr}\,[X_\mu, X_\nu]^2$

expand around the classical solution

$$\begin{cases} X_1 = \hat{q} \\ X_2 = \hat{p} \end{cases} \tag{42}$$

$\Longrightarrow$ NCYM (Aoki-Ishibashi-Iso-Kawai-Kitazawa-Tada '99)

$[\hat{q}, \hat{p}] = i$ realizable only at $N = \infty$

finite N version:

Eguchi-Kawai model

$$S = -\mathcal{Z}_{\mu\nu}\mathrm{Tr}(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger) \tag{43}$$

$$\text{classical solution : } \begin{cases} U_1^{(0)} = Q \\ U_2^{(0)} = P \end{cases} \tag{44}$$

$U_\mu = U'_\mu U_\mu^{(0)} \Longrightarrow$ NCYM on the lattice

(Ambjørn-Makeenko-J.N.-Szabo '99)

nonperturbative studies of various field theories on NCG

(Bietenholz-Hofheinz-J.N. '02, Ambjørn-Catterall '02)

6. Summary and Discussions

- Deconstruction
  - creating new dimensions (latticized) from internal d.o.f.
  - naturally realizable in matrix models "orbifolding"
  - toroidal compactifications of M-theory

- a proposal for SYM on the lattice
  - moduli instability ?
  - presence of gravity

- matrix models and NC geometry
  - dynamical generation of 4d space-time
    - New Monte Carlo approach
    - Gaussian expansion method
  - emergence of local field theory, chiral fermions, gauge group, etc.