Ginsparg-Wilson 関係式 と
カイラルゲージ理論のゲージ不変な構成

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Plan of this talk:

1. Introduction
2. Ginsparg-Wilson 関係式と格子上の Weyl fermion
3. Reconstruction theorem と 4+2 次元 local cohomology problem
   - SU(2) x U(1) における解析
4. U(1) カイラルゲージ理論の具体的構成
5. Nicolai Mapping と Ginsparg-Wilson 関係式

Based on

- Y. Kikukawa, Y. Nakayama,
  "Gauge anomaly cancellations in $\mathrm{SU}(2)_{\boldsymbol{L}}\times\mathrm{U}(1)_{\boldsymbol{Y}}$ electroweak theory on the lattice",
  Nucl. Phys. B597 (2001) 519
- D. Kadoh, Y. Kikukawa, Y. Nakayama, in preparation
- Y. Kikukawa, Y. Nakayama,
  "Nicolai mapping vs. exact chiral symmetry on the lattice",
  Phys. Rev. D66 (2002) 094508

Why chiral gauge theories on the lattice ?

- Chiral gauge theories
  - Gauge symmetry prohibits the mass terms of fermions
  - Gauge anomaly cancellation
- Examples of chiral gauge theories (anomaly-free)
  - standard model: SU(3)×SU(2)×U(1) gauge theory
  - Georgi-Glashow grand unified model : SU(5) gauge theory (SO(10))
  - (Extended) Technicolor model
  - ...
- (Possible) Dynamics of chiral gauge theories
  - Dynamical gauge symmetry breaking (MAC: Most attractive channel)
  - Massless composite fermions ('t Hooft anomaly matching condition)

Spontaneous breakdown of electroweak gauge symmetry

- need better understanding of the mechanism
- dynamics of chiral gauge theories can be relevant

$\Longleftarrow$ non-perturbative formulation
$\Longrightarrow$ in the framework of lattice gauge theory (cf. Lattice QCD)

Recent development
Gauge-covariant and local lattice Dirac operator
satisfying the Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

P. Hasenfratz et at., Neuberger, Hernández et al.

$\Longrightarrow$ Exact chiral symmetry on the lattice
$\Longrightarrow$ Gauge-invariant construction of chiral gauge theories on the lattice Lüscher

- Abelian chiral gauge theories

$\Longleftarrow$ an approach using domain wall fermion
towards a practical (numerical) implementation

Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

In terms of the fermion propagator $S_F = D^{-1}$:

$$\gamma_5 S_F(x,y) + S_F(x,y)\gamma_5 = \gamma_5 2\delta(x,y)$$

- chiral symmetry is broken only in local contact terms
- such local terms do not contribute to the physical amplitudes evaluated at long-distance, $x - y \neq 0$
- represents the chiral limit for lattice Dirac operators in a consistent manner with Nielsen-Ninomiya theorem

Exact Chiral Symmetry on the Lattice

Ginsparg-Wilson relation implies an exact symmetry of the fermion action ! Lüscher

Under the following transformation,

$$\delta\psi(x) = \gamma_5 (1 - 2aD)\,\psi(x), \quad \delta\bar\psi(x) = \bar\psi(x)\gamma_5$$

the fermion action is invariant,

$$\delta S = a^4 \sum_x \bar\psi \left\{ D\gamma_5 (1 - 2aD) + \gamma_5 D \right\} \psi(x) = 0$$

- local transformation as long as D is local
- can be regarded as the lattice counterpart of chiral symmetry in the continuum theory

Structure of Ginsparg-Wilson fermion

Locality of D $\Rightarrow$	GW rel. $\Rightarrow$	Index of D
	$\gamma_5 D + D\gamma_5 = 2D\gamma_5 D$	$\Downarrow$
$\Uparrow$	$\searrow$ Local chiral anomaly	Index theorem on the lattice
	$\sum_x -\mathrm{tr}\gamma_5 D(x,x)$ $\searrow$	$\Uparrow$
Admissibility cond. $\Rightarrow$	Topological structure of lattice gauge fields $\Rightarrow$	Topological charge
$\|1 - P_{\mu\nu}(x)\| < \epsilon$		$Q = -\frac{1}{2}\sum_x \mathrm{tr}\gamma_5 D(x,x)$

the Ginsparg-Wilson relation:

$$\hat{\gamma}_5 \equiv \gamma_5 (1 - 2aD)$$

$$\gamma_5 D + D\hat{\gamma}_5 = 0$$

$$\{\hat{\gamma}_5\}^2 = 1$$

Weyl fermion:

$$\left(\frac{1+\hat{\gamma}_5}{2}\right)\psi_R(x) = \psi_R(x), \qquad \bar{\psi}_R(x)\left(\frac{1-\gamma_5}{2}\right) = \bar{\psi}_R(x)$$

Path-Integral measure $\Longleftrightarrow$ Chiral bases: $\{v_i(x)\}$ and $\{\bar{v}_k(x)\}$

$$\hat{P}_+ v_i(x) = v_i(x), \qquad \hat{P}_+ = \left(\frac{1+\hat{\gamma}_5}{2}\right)$$

$$\bar{v}_k(x) P_- = \bar{v}_k(x), \qquad P_- = \left(\frac{1-\gamma_5}{2}\right)$$

$$\psi_R(x) = \sum_i v_i(x) c_i, \quad \bar{\psi}_R(x) = \sum_k \bar{c}_k \bar{v}_k(x)$$

$$\mathcal{D}[\psi_R]\mathcal{D}[\bar{\psi}_R] \equiv \prod_i dc_i \prod_k d\bar{c}_k$$

change of the chiral basis by a unitary transformation:

$$\tilde{v}_i(x) = v_j(x)\left(\tilde{Q}^{-1}\right)_{ji}$$

$$\tilde{c}_i = \tilde{Q}_{ij} c_j$$

$$\mathcal{D}[\psi_R]\mathcal{D}[\bar{\psi}_R] \Longrightarrow \mathcal{D}[\psi_R]\mathcal{D}[\bar{\psi}_R]\det\tilde{Q}$$

• Chiral determinant or effective action

$$e^{\Gamma_{\text{eff}}(U_\mu)} = \int \mathcal{D}[\psi_R]\mathcal{D}[\bar\psi_R]\; e^{\sum_x \bar\psi_R(x) D \psi_R(x)}$$
$$= \int \prod_i dc_i \prod_k d\bar c_k \; e^{\sum_{k,i} \bar c_k (\bar v_k, D v_i) c_i}$$
$$= \det\left(\bar v_k D v_j\right)$$

cf. Overlap Formula

• Variation of effective action with respect to gauge field

$$U_\mu(x) \longrightarrow U_\mu(x) + \delta_\eta U_\mu(x), \qquad \delta_\eta U_\mu(x) = \eta_\mu(x) U_\mu(x)$$

$$\delta_\eta \Gamma_{\text{eff}} = \delta_\eta \mathrm{Tr}\,\mathrm{Ln}\,(\bar v_k, D v_i)$$
$$= \{(\bar v_k, (\delta_\eta D) v_i) + (\bar v_k, D(\delta_\eta v_i))\} \left\{(\bar v, D v)^{-1}\right\}_{ik}$$
$$= \mathrm{Tr}\left\{(\delta_\eta D)\hat P_+ D^{-1} P_-\right\} + \sum_i (v_i, \delta_\eta v_i)$$

$$\left\{(\bar v, D v)^{-1}\right\}_{ik} = v_i^\dagger \hat P_+ D^{-1} P_- \bar v_k$$

$$\hat P_+ = \sum_i v_i \otimes v_i^\dagger, \quad P_- = \sum_k (\bar v_k)^\dagger \otimes \bar v_k$$

$$\mathfrak{L}_\eta = i \sum_i (v_i, \delta_\eta v_i) \equiv a^4 \sum_x \eta_\mu(x) j_\mu(x)$$

• Field equation (S-D equation)

$$\begin{aligned} 0 &= \int [dU_\mu]\, U_\mu(x) \frac{\delta}{\delta U_\mu(x)} \left\{ e^{-S_G[U]} \det(\bar{v}_k D v_j) O[U] \right\} \\ &= \langle \delta_x O[U] \rangle + \langle \{ -\delta_x S_G[U] + \delta_x \Gamma_{\text{eff}}[U] \}\, O[U] \rangle \end{aligned}$$

$$\delta_x \Gamma_{\text{eff}} = \langle \sum_y \bar{\psi}_R(y) \delta_x D \psi_R(y) \rangle - i j_\mu(x)$$

local field equation $\Longrightarrow$ local $j_\mu(x)$

• Gauge anomaly

$$\eta_\mu(x) = -i \nabla_\mu \omega(x), \qquad \delta_\eta D = i[\omega, D]$$

$$\begin{aligned} \delta_\eta \Gamma_{\text{eff}} &= i \text{Tr} \omega \left(P_- - \hat{P}_+ \right) + \sum_i (v_i, \delta_\eta v_i) \\ &= i \text{Tr} \omega \gamma_5 \left(1 - \frac{a}{2} D \right) + \sum_x \omega(x) \cdot \nabla^*_\mu j_\mu(x) \end{aligned}$$

In the continuum limit

$$i \text{tr} T^a \gamma_5 \left(1 - \frac{a}{2} D \right)(x, x) \simeq \frac{i}{64\pi^2} d^{abc} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^b_{\rho\sigma}(x) + \mathcal{O}(a)$$

$$d^{abc} = \text{Tr} T^a \{ T^b, T^c \}$$

Exact gauge invariance $\Longrightarrow$

$$\sum_r d^{abc} = 0, \qquad \left\{ \nabla^*_\mu j_\mu(x) \right\}^a = -i \text{tr} T^a \gamma_5 \left(1 - \frac{a}{2} D \right)(x, x)$$

- Integrability condition:

(local version)

$$\begin{aligned}\delta_\zeta \mathfrak{L}_\eta - \delta_\eta \mathfrak{L}_\zeta + a\mathfrak{L}_{[\zeta,\eta]} &= i\sum_i (\delta_\zeta v_i, \delta_\eta v_i) - (\zeta \leftrightarrow \eta) \\ &= i\mathrm{Tr}\, \hat{P}_+ \left[\delta_\zeta \hat{P}_+, \delta_\eta \hat{P}_+\right]\end{aligned}$$

(global version)

$$\exp\left(i \oint dt\, \mathfrak{L}_\eta\right) = \det(1 - P_0 + P_0 Q_1)$$

$$U_\mu(x;t) \quad t \in [0,1] \text{ (a closed loop)}$$

$$P_t \equiv \hat{P}_+(t)$$

$$\begin{aligned}P_t = Q_t P_0 Q_t^{-1} \qquad & Q_t : \text{Unitary} \\ \partial_t Q_t = [\partial_t P_t, P_t] Q_t \qquad & Q_0 = 1\end{aligned}$$

$$\begin{aligned}v_i &= Q_t \sum_l v_l|_{t=0} (\mathcal{S}^{-1})_{li}, \ \mathcal{S}|_{t=0} = 1 \\ \mathfrak{L}_\eta &= i\sum_i (v_i, \partial_t v_i) \\ &= i\sum_l (v_l|_{t=0}, Q_t^{-1}\partial_t Q_t v_l|_{t=0}) + i\sum_{i,l} \mathcal{S}_{il}\partial_t(\mathcal{S}^{-1})_{li} \\ &= -i\partial_t \ln\det \mathcal{S}\end{aligned}$$

$$\therefore \ \exp\left\{i\int_0^1 dt \mathfrak{L}_\eta\right\} = \det \mathcal{S}|_{t=1}$$

Construction of the Path-Integral measure

$$\mathcal{D}[\psi_R]\mathcal{D}[\bar{\psi}_R] \equiv \prod_i dc_i \prod_k d\bar{c}_k \; (U_\mu(x))$$

with the properties:

1. Local field equation (S-D equation)
2. Gauge invariance at finite lattice spacing
3. Smooth dependence on $\{U_\mu(x)\}$

$\Longrightarrow$

Construction of the measure term (current)

$$\mathfrak{L}_\eta = i \sum_i (v_i, \delta_\eta v_i) \equiv a^4 \sum_x \eta_\mu(x) j_\mu(x)$$

with the properties:

1. local with respect to $\{U_\mu(x)\}$
2. anomalous conservation law

$$\left\{ \nabla_\mu^* j_\mu(x) \right\}^a = -i \operatorname{tr} T^a \gamma_5 \left(1 - \frac{a}{2} D \right)(x, x)$$

3. global integrability condition

$$\exp\left(i \oint dt \, \mathfrak{L}_\eta \right) = \det(1 - P_0 + P_0 Q_1) \quad (\eta = \partial_t U \cdot U^{-1})$$

- Topological properties of gauge anomalies on the lattice

  – Four-dimensional lattice plus two continuum dimensions

$$U_\mu(z), \qquad z = (x_\mu, s, t)$$

$$\eta_\mu = \partial_s U_\mu \cdot U_\mu^{-1}, \quad \zeta_\mu = \partial_t U_\mu \cdot U_\mu^{-1}$$

$$q(z) \equiv i\mathrm{tr}\, \hat{P}_+ \left[\partial_t \hat{P}_+, \partial_s \hat{P}_+\right](x, x; s, t)$$

$$q(z) \simeq -\frac{1}{32\pi^2} d^{abc}\, \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(z) \partial_s A^b_\rho(z) \partial_t A^c_\sigma(z) + \mathcal{O}(a)$$

  –

$$\sum_x \int dsdt\, \delta q(z) = 0$$

$q(z)$ is a 4+2 dimensional local topological field

cf. Alvarez-Gaumé and Ginsparg

$$\delta \int dsdt\ i\mathrm{Tr}\hat{P}_+ \left[\partial_t \hat{P}_+, \partial_s \hat{P}_+\right]$$
$$= \int dsdt\ \left\{ i\mathrm{Tr}\delta\hat{P}_+ \left[\partial_t \hat{P}_+, \partial_s \hat{P}_+\right] + i\mathrm{Tr}\hat{P}_+ \left[\partial_t \delta\hat{P}_+, \partial_s \hat{P}_+\right] + i\mathrm{Tr}\hat{P}_+ \left[\partial_t \hat{P}_+, \partial_s \delta\hat{P}_+\right] \right\}$$

$$\mathrm{Tr}\delta_1\hat{P}_+ \cdot \delta_2\hat{P}_+ \cdot \delta_3\hat{P}_+ \simeq \mathrm{Tr}\delta_1\hat{\gamma}_5 \cdot \delta_2\hat{\gamma}_5 \cdot \delta_3\hat{\gamma}_5 = 0$$
$$(\hat{\gamma}_5)^2 = 1$$
$$\delta\hat{\gamma}_5 \cdot \hat{\gamma}_5 + \hat{\gamma}_5 \cdot \delta\hat{\gamma}_5 = 0$$

$$= \int dsdt \left\{ \partial_s i\mathrm{Tr}\hat{P}_+ \left[\partial_t \hat{P}_+, \delta\hat{P}_+\right] - \partial_t i\mathrm{Tr}\hat{P}_+ \left[\delta_t \hat{P}_+, \partial_s \delta\hat{P}_+\right] \right\}$$
$$= 0$$

1. Local cohomology problem

- 4+2 dimensional gauge invariance

$$D_{s,t}U_\mu(z) \equiv \partial_{s,t}U_\mu(z) + A_{s,t}(z)U_\mu(z) - U_\mu(z)A_{s,t}(z+\hat{\mu})$$

$$\sum_x q(z) = i\mathrm{Tr}\left\{\hat{P}_+\left[\partial_t\hat{P}_+, \partial_s\hat{P}_+\right] - \frac{1}{2}\partial_t[A_s\hat{\gamma}_5] + \frac{1}{2}\partial_s[A_t\hat{\gamma}_5]\right\}$$

- if $q(z)$ is cohomologically trivial

$$q(z) = \partial^*_\mu k_\mu(z) + \partial_t k_s(z) - \partial_s k_t(z)$$

where (k_μ, k_s, k_t) are gauge-invariant and local currents

$\Longrightarrow$

$$\sum_x k_{s,t}(z) = \sum_y D_{s,t}U_\mu \cdot U_\mu^{-1}(w)\; j_\mu(w)$$

where $z = (x, s, t)$ and $w = (y, s, t)$

$$i\mathrm{Tr}\left\{\hat{P}_+\left[\partial_t\hat{P}_+, \partial_s\hat{P}_+\right] - \frac{1}{2}\partial_t[A_s\hat{\gamma}_5] + \frac{1}{2}\partial_s[A_t\hat{\gamma}_5]\right\}$$
$$= \partial_t\sum_y\left[D_sU_\mu U_\mu^{-1}(w)j_\mu(w)\right] - \partial_s\sum_y\left[D_tU_\mu U_\mu^{-1}(w)j_\mu(w)\right]$$

- anomalous conservation law : set $U_\mu(z) = U_\mu(x)$, not depend on s, t

$$\nabla^*_\mu j_\mu(x) = -i\mathrm{tr}\gamma_5\left(1 - \frac{a}{2}D\right)(x,x)$$

- local integrability condition : set $A_{s,t}(z) = 0$

$$\delta_\zeta\mathcal{L}_\eta - \delta_\eta\mathcal{L}_\zeta \;=\; i\mathrm{Tr}\,\hat{P}_+\left[\delta_\zeta\hat{P}_+, \delta_\eta\hat{P}_+\right]$$

where $\eta_\mu = \partial_s U_\mu U_\mu{}^{-1}$ and $\zeta_\mu = \partial_t U_\mu U_\mu{}^{-1}$

2. Reconstruction of the fermion measure

$$U_\mu(x;t) \quad t \in [0,1] \quad (\text{interpolation !})$$

$$v_i(x) = \begin{cases} Q_1 w_1(x) W^{-1} \\ Q_1 w_i(x) & (i \neq 1) \end{cases}$$

$$W \equiv \exp\left\{ i \oint_0^1 dt \mathfrak{L}_\eta \right\}$$

Measure so defined is independent on the path of the interpolation

$$\begin{aligned} \tilde{v}_i(x) &= \begin{cases} \tilde{Q}_1 w_1(x) \tilde{W}^{-1} \\ \tilde{Q}_1 w_i(x) & (i \neq 1) \end{cases} \\ &= \begin{cases} \tilde{Q}_1 Q_1^{-1} v_1(x) W \tilde{W}^{-1} \\ \tilde{Q}_1 Q_1^{-1} v_i(x) & (i \neq 1) \end{cases} \end{aligned}$$

but for the closed loop

$$\tilde{W} W^{-1} = \det\left(1 - P_1 + P_1 \tilde{Q}_1 Q_1^{-1}\right)$$

$$\mathcal{D}[\psi_R]\mathcal{D}[\bar{\psi}_R] \Longrightarrow \mathcal{D}[\psi_R]\mathcal{D}[\bar{\psi}_R] \times \det\left(1 - P_1 + P_1 \tilde{Q}_1 Q_1^{-1}\right) \tilde{W}^{-1} W$$

$$q(z) = q(z)\left[U_\mu^{(2)}, U_\mu^{(1)}\right]$$

Cohomological analysis of $U_\mu^{(1)}$ $\Longleftarrow$ Poincaré lemma on the lattice
cf. Lüscher, Fujiwara-Suzuki-Wu

$$\begin{aligned} q(z) \;=\; & \alpha\left[U_\mu^{(2)}\right] + \beta_{\mu\nu}\left[U_\mu^{(2)}\right] F_{\mu\nu}^{(1)}(z) \\ & + \gamma_{\mu\nu\rho\sigma}\left[U_\mu^{(2)}\right] F_{\mu\nu}^{(1)}(z) F_{\rho\sigma}^{(1)}(z+\hat\mu+\hat\nu) \\ & + \delta\epsilon_{\mu\nu\rho\sigma\lambda\tau} F_{\mu\nu}^{(1)}(z) F_{\rho\sigma}^{(1)}(z+\hat\mu+\hat\nu) \times \\ & \qquad F_{\lambda\tau}^{(1)}(z+\hat\mu+\hat\nu+\hat\rho+\hat\sigma) \\ & + \partial_\mu^* k_\mu(z) \end{aligned}$$

where

$$F_{\mu\nu}^{(1)}(z) = \partial_\mu A_\nu(z) - \partial_\nu A_\mu(z) \qquad (\mu, \nu, \cdots = 1, 2, \cdots, 6)$$

$$\partial_\mu^* \beta_{\mu\nu}(z) = 0, \quad \partial_\mu^* \gamma_{\mu\nu\rho\sigma}(z) = 0$$

- Admissible gauge field $(k, l = 1, 2, 3, 4)$

$$\| 1 - P_{kl}^{(1)} \| < \epsilon < \frac{1}{3}\pi$$

$$\left(\epsilon_{klmn}\partial_l F_{mn}^{(1)}(z) = 0\right)$$

$\Longrightarrow$ Parameterization by the vector potential

$$U_k^{(1)}(z) = \exp\left(i A_k(z)\right)$$

$$F_{kl}^{(1)} \equiv \frac{1}{i}\ln P_{kl}^{(1)} = \partial_k A_l(z) - \partial_l A_k(z)$$

• Special properties of $q(z)$ in the electroweak theory

$$q(z)[U_\mu^{(2)}, A_\mu] = -q(z)[(U_\mu^{(2)})^*, -A_\mu] = -q(z)[U_\mu^{(2)}, -A_\mu]$$

$$q(z)[U_\mu^{(2)}, 0] = 0$$

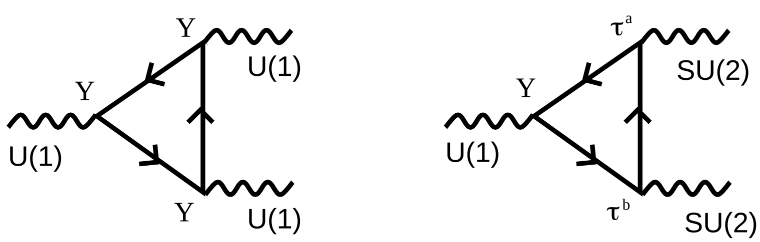
$\Longrightarrow$

$$\begin{aligned} q(z) \;=\; & \beta_{\mu\nu}\left[U_\mu^{(2)}\right] F_{\mu\nu}^{(1)}(z) \\ & +\delta\epsilon_{\mu\nu\rho\sigma\lambda\tau} F_{\mu\nu}^{(1)}(z) F_{\rho\sigma}^{(1)}(z+\hat{\mu}+\hat{\nu}) \times \\ & \qquad\qquad F_{\lambda\tau}^{(1)}(z+\hat{\mu}+\hat{\nu}+\hat{\rho}+\hat{\sigma}) \\ & +\partial_\mu^* k_\mu(z) \end{aligned}$$

• Anomaly cancellation conditions in the electroweak theory

$$\begin{pmatrix} \nu_L(x) \\ e_L(x) \end{pmatrix}_{Y=-\frac{1}{2}}, \quad e_R(x)_{Y=-1}, \quad \begin{pmatrix} u_{L\,i}(x) \\ d_{L\,i}(x) \end{pmatrix}_{Y=\frac{1}{6}}, \quad \begin{matrix} u_{R\,i}(x)_{Y=+\frac{2}{3}} \\ d_{R\,i}(x)_{Y=-\frac{1}{3}} \end{matrix}$$

where i is the color index $(i = 1, 2, 3)$.



$$\sum_L Y^3 - \sum_R Y^3 = 0$$

$$\sum_{\text{doublet(L)}} Y = 0$$

$$\sum_{\text{singlet(R)}} Y = 0$$

Summary(1)

So far

Complete construction:

1. **Abelian chiral gauge theories (in finite volume)** Lüscher

Gauge anomaly cancellation:

1. Non-abelian chiral gauge theories in all orders of lattice perturbation theory Suzuki, Lüscher
2. $SU(2)_L \times U(1)_Y$ electroweak theory Nakayama-Y.K.

Global aspects:

1. SU(2) doublet Neuberger, Bär-Campos

- Anomaly-free condition

$$\sum_{\alpha=1}^{N} e_\alpha^3 = 0$$

- On a finite lattice of the size L with P.B.C.

$$\Gamma = \{x = (x_0, \cdots, x_3) \in \mathbb{Z}^4 \mid 0 \leq x_\mu < L\}$$

$$U(x + L\hat{\nu}, \mu) = U(x, \mu)$$
$$\psi_L(x + L\hat{\mu}) = \psi_L(x)$$

- admissible U(1) gauge fields

$$\begin{aligned} F_{\mu\nu}(x) &= \frac{1}{i} \ln P(x, \mu, \nu) \\ P(x, \mu, \nu) &= U(x, \mu) U(x + \hat{\mu}, \nu) U(x + \hat{\nu}, \mu)^{-1} U(x, \nu)^{-1} \end{aligned}$$

$$|F_{\mu\nu}(x)| < \epsilon$$

– Mangetic flux sectors

$$\phi_{\mu\nu}(x) = \sum_{s,t=0}^{L-1} F_{\mu\nu}(x + s\hat{\mu} + t\hat{\nu}) = 2\pi m_{\mu\nu}$$

- Dirac operator (kernel) : localization range ϱ

$$D\psi(x) = \sum_{y \in \mathbb{Z}^4} D(x, y) \psi(y)$$

$$D\psi(x) = \sum_{y \in \Gamma} D_L(x, y) \psi(y), \quad D_L(x, y) = \sum_{n \in \mathbb{Z}^4} D(x, y + nL)$$

$$D_L(x, y) = D(x, y) + \mathrm{O}\left(\mathrm{e}^{-L/\varrho}\right)$$

The space of admissible U(1) gauge fields

- a unique parametrization of $U(x,\mu)$

$$U(x,\mu) = \tilde{U}(x,\mu)\, V_{[m]}(x,\mu)$$

$$V_{[m]}(x,\mu) = \mathrm{e}^{-\frac{2\pi i}{L^2}\left[L\delta_{\tilde{x}_\mu,L-1}\sum_{\nu>\mu} m_{\mu\nu}\tilde{x}_\nu + \sum_{\nu<\mu} m_{\mu\nu}\tilde{x}_\nu\right]}$$

$$(\tilde{x}_\mu = x_\mu \bmod L)$$

$$\tilde{U}(x,\mu) \;=\; \mathrm{e}^{iA_\mu^T(x)}\, U_{[w]}(x,\mu)\, \Lambda(x)\Lambda(x+\hat{\mu})^{-1}$$

$$A_\mu^T(x) \;=\; \sum_{y\in\Gamma_d} G_L(x-y)\partial_\lambda^* F_{\lambda\mu}(y)$$

$$U_{[w]}(x,\mu) \;=\; \begin{cases} w_\mu & \text{if } x_\mu = 0 \bmod \mathrm{L} \\ 1 & \text{otherwise} \end{cases}$$

$$\partial_\mu^*\partial_\mu G_L(z) = \delta_{\tilde{z},0} - L^{-d}$$
$$G_L(z+L\hat{\mu}) = G_L(z)$$
$$\sum_{z\in\Gamma_d} G_L(z) = 0$$

- Topology of the space

$$\mathfrak{U}[m] \cong \mathrm{U}(1)^4 \times \mathrm{U}(1)^{L^4} \times \mathfrak{A}[m]$$

U(1) bundle associated with the measure

- A U(1) bundle (determinant line bundle) associated with the basis transformation

$$v_i^b = v_j^a Q_{ji}(a \longrightarrow b)$$

$$g_{ab} = \det Q_{ji}(a \longrightarrow b)$$

$$g_{ab} \longrightarrow h_a \, g_{ab} \, h_b^{-1}$$

  – the connection of the U(1) bundle: "measure term"

$$\mathfrak{L}_\eta^a = i \sum_i \left(v_i^a, \delta_\eta v_i^a\right)$$

$$\mathfrak{L}_\eta^b = \mathfrak{L}_\eta^a - i\delta_\eta \ln\det Q(a \longrightarrow b)$$

  – the curvature of the U(1) bundle: "integrability condition (local)"

$$\delta_\zeta \mathfrak{L}_\eta^a - \delta_\eta \mathfrak{L}_\zeta^a = i\mathrm{Tr}_L \, \hat{P}_- \left[\delta_\zeta \hat{P}_-, \delta_\eta \hat{P}_-\right]$$

  – the Wilson line of the U(1) bundle: "integrability condition (global)"

$$\exp\left(i \oint_{T^1} dt \, \mathfrak{L}_\eta^a\right) = \det(1 - \hat{P}_0 + \hat{P}_0 Q_1)$$

- Smooth measure over $\mathfrak{U}[m]$
  – if and only if the U(1) bundle is trivial
  – the U(1) bundle on T^n is trivial if the magnetic flux vanishes:

$$\int_{T^2} i\mathrm{Tr}_L \, \hat{P}_- \left[\delta_\zeta \hat{P}_-, \delta_\eta \hat{P}_-\right] = 0$$

- Non-trivial U(1) bundle ⇔ Global obstruction along a gauge loop: Gauge anomaly ⇒ Global obstruction

Alvarez-Gaumé and Ginsparg

Global construction of the measure term

$$\mathfrak{L}_\eta = \sum_x \eta_\mu(x) j_\mu(x) \quad : \quad \text{smooth on } \mathfrak{U}[m]$$

- Integrability condition (local) $\Longrightarrow$ magnetic flux vanished !

$$\delta_\zeta \mathfrak{L}_\eta - \delta_\eta \mathfrak{L}_\zeta = i \mathrm{Tr}_L \, \hat{P}_- \left[\delta_\zeta \hat{P}_-, \delta_\eta \hat{P}_- \right]$$

$$g_{ab} = 1 \qquad \text{(the U(1) bundle is trivial)}$$

- Integrability condition (global) $\Longrightarrow$ the Wilson line reproduced

$$\exp\left(i \oint_{T^1} dt \, \mathfrak{L}_\eta \right) = \det(1 - \hat{P}_0 + \hat{P}_0 Q_1)$$

$$\mathfrak{L}^a_\eta = \mathfrak{L}_\eta - i \delta_\eta \ln \det Q$$

(U(1) bundles are equivalent)

- Gauge anomaly cancellation

  for $\eta_\mu(x) = -\partial_\mu \omega(x)$

$$\mathfrak{L}_\eta = \sum_x \omega(x) \partial^*_\mu j_\mu(x) = \sum_x \omega(x) \mathrm{tr} \, (1 - aD) \, (x, x)$$

• Use of the infinite lattice

$$\mathfrak{L}_\eta = \mathfrak{L}^\star_\eta + \Delta\mathfrak{L}_\eta$$

$$i\mathrm{Tr}\,\hat{P}_-\left[\delta_\zeta\hat{P}_-,\delta_\eta\hat{P}_-\right] = i\mathrm{Tr}\,\hat{P}_-\left[\delta_\zeta\hat{P}_-,\delta_\eta\hat{P}_-\right] + i\Delta\mathrm{Tr}\,\hat{P}_-\left[\delta_\zeta\hat{P}_-,\delta_\eta\hat{P}_-\right]$$

$$q_L(x) = q(x) + \Delta q(x)$$

$$\text{cf.}\quad D_L(x,y) = D(x,y) + \mathrm{O}\left(\mathrm{e}^{-L/\varrho}\right)$$

$$\begin{aligned}\mathfrak{L}^\star_\eta &= i\int_0^1 dt\,\mathrm{Tr}_L\left\{\hat{P}_-[\partial_t\hat{P}_-,\delta_\eta\hat{P}_-]\right\} + \\ &\quad \int_0^1 dt\sum_{x\in\Gamma_4}\{\eta_\mu(x)\bar{k}_\mu(x) + A_\mu(x)\delta_\eta\bar{k}_\mu(x)\}\end{aligned}$$

$$U_t(x,\mu) = \mathrm{e}^{itA_\mu(x)}$$

• Use of vector potential

$$\begin{aligned}&U(x,\mu) = \mathrm{e}^{iA_\mu(x)},\quad |A_\mu(x)| \leq \pi(1 + 8 \parallel x \parallel)\\ &F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\end{aligned}$$

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\omega(x)$$

• Cohomological analysis in the infinite lattice

$$q(x) = \mathrm{tr}\,(1 - aD)\,(x,x)$$

$$\sum_x \delta q(x) = 0$$

$$q(x) = \gamma\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}(x)\,F_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) + \partial^*_\mu\bar{k}_\mu(x)$$

Lüscher, Fujiwara-Suzuki-Wu

- $$U_t(x,\mu) = \mathrm{e}^{itA_\mu(x)}$$

$$\begin{aligned} q(x) &= \alpha + \int_0^1 dt \sum_y \left.\frac{\partial q(x)}{\partial A_\mu(y)}\right|_{A\to tA} A_\mu(y) \\ &= \alpha + \sum_y J_\mu(x,y) A_\mu(y) \end{aligned}$$

- Topological property and Gauge invariance

$$\sum_x J_\mu(x,y) = 0$$

$$J_\mu(x,y)\overleftarrow{\partial}^*_\mu = 0$$

- Poincaré lemma

  Lemma

  f be a k-form which satisfies

$$d^* f = 0 \quad \text{and} \quad \sum_{x\in\Gamma_n} f(x) = 0 \ \text{ if } \ k = 0.$$

  Then there exist a form $g \in \Omega_{k+1}$ such that

$$f = d^* g$$

A practical implementation of U(1) Chiral Gauge Theories

Kadoh-Nakayama-Y.K.

- A finite algorithm to produce the Weyl fermion measure for a given admissible U(1) gauge fields
  - Gauge invariance
  - Locality
  - Smoothness
- Issues in practical implementations
  - Use of the infinite lattice
  - Continuous interpolation

$$U_t(x, \mu) = \mathrm{e}^{itA_\mu(x)}$$

  - Vector potential $A_\mu(x)$ is not bounded
- Our approach
  - U(1) construction within a finite lattice
  - Discrete interpolation within the Z_N subspace

$\Longrightarrow$ Z_N chiral gauge theories with exact Z_N gauge invariance

- Poincaré lemma on a finite lattice

$$\Gamma_n = \{x \in \mathbb{Z}^n | -L/2 \le x_\mu < L/2\}$$

anti-symmetric tensor fields

$$f_{\mu_1\cdots\mu_k}(x + L\hat{\nu}) = f_{\mu_1\cdots\mu_k}(x) \ \ (\text{for all } \mu,\nu = 1,\cdots,n)$$

locality properties

a reference point $x_0 \in \Gamma_n$

$-L/2 \le (x_\mu - x_{0\mu}) < L/2$ (mod L)

$$\begin{aligned} |f_{\mu_1\cdots\mu_k}(x)| &< C_1(1+ \| x - x_0 \|^{p_1})\, \mathrm{e}^{-\|x-x_0\|/\varrho} && (\| x - x_0 \| < L/2) \\ |f_{\mu_1\cdots\mu_k}(x)| &< C_2 L^{p_2}\, \mathrm{e}^{-L/2\varrho} && (\| x - x_0 \| \ge L/2) \end{aligned}$$

Lemma a Let f be a k-form which satisfies

$$d^* f = 0 \quad \text{and} \quad \sum_{x\in\Gamma_n} f(x) = 0 \ \text{ if } \ k = 0.$$

Then there exist a form $g \in \Omega_{k+1}$ and a form $\Delta f \in \Omega_k$ such that

$$f = d^* g + \Delta f, \qquad |\Delta f_{\mu_1\cdots\mu_k}(x)| < c L^\sigma \mathrm{e}^{-L/2\varrho}$$

Lemma b Let f be a k-form which satisfies

$$d^* f = 0 \ \text{ and } \ \sum_{x\in\Gamma_n} f(x) = 0.$$

Then there exist a form $g \in \Omega_{k+1}$ such that

$$f = d^* g$$

- Bounded vector potential $A_\mu(x)$

$$U(x,\mu) = \tilde{U}(x,\mu)\, V_{[m]}(x,\mu)$$

$$\begin{aligned} & e^{i\tilde{A}_\mu(x)} = \tilde{U}(x,\mu) \\ & \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) = F_{\mu\nu}(x) - \frac{2\pi m_{\mu\nu}}{L^2} \\ & \begin{cases} |\tilde{A}_\mu(x)| \leq \pi(1 + 4 \parallel x \parallel) & x_\nu \neq L/2 - 1 \\ |\tilde{A}_\mu(x)| \leq \pi(1 + 2L + 6L^2) & \text{otherwise} \end{cases} \end{aligned}$$

$$\tilde{A}'_\mu(x) = \tilde{A}_\mu(x) + \partial_\mu \omega(x)$$

- Interpolation with $\tilde{A}_\mu(x)$

$$U_t(x,\mu) = e^{it\tilde{A}_\mu(x)}\, V_{[m]}(x,\mu)$$

- Cohomological analysis within a finite lattice

$$q_L(x) = \mathrm{tr}\,(1 - aD_L)\,(x,x)$$

$$\sum_x \delta q_L(x) = 0$$

$$\begin{aligned} q_L(x) \;&=\; q_{[m]}(x) + \phi_{[m]\mu\nu}(x)\, \tilde{F}_{\mu\nu}(x) \\ &+\; \gamma_{[m]}\epsilon_{\mu\nu\rho\sigma}\tilde{F}_{\mu\nu}(x)\, \tilde{F}_{\rho\sigma}(x + \hat{\mu} + \hat{\nu}) + \partial^*_\mu \tilde{k}_\mu(x) \end{aligned}$$

- the measure term on a finite lattice

$$\begin{aligned} \mathfrak{L}_\eta \;=\; & i \int_0^1 dt\, \mathrm{Tr}_L \left\{ \hat{P}_-[\partial_t \hat{P}_-, \delta_\eta \hat{P}_-] \right\} + \\ & \int_0^1 dt \sum_{x\in\Gamma_4} \left\{ \eta_\mu(x)\bar{k}_\mu(x) + \tilde{A}_\mu(x)\delta_\eta \bar{k}_\mu(x) \right\} \end{aligned}$$

- the $\boldsymbol{Z_N}$ subspace

$$U(x,\mu) \in Z_N \qquad (N = pL^2 \quad p \in \mathbb{Z})$$

$$U(x,\mu) = \mathrm{e}^{i\tilde{A}_\mu(x)}\, V_{[m]}(x,\mu)$$

$$\tilde{A}_\mu(x) = \frac{2\pi}{N}\left(\bar{q}_\mu(x) + \partial_\mu z(x)\right)$$

$$\bar{q}_\mu(x) = \frac{N}{2\pi}\left(A^T_\mu(x) - i \ln w_\mu \delta_{x_\mu,0}\right)$$

- Interpolation within the $\boldsymbol{Z_N}$ subspace

  –

$$t^{(s)} = \frac{s}{M} \quad (s = 0, 1, \cdots, M) \qquad M \equiv \mathrm{Max}_{x,\mu}\left\{|\bar{q}_\mu(x)|\right\}$$

$$\tilde{A}_\mu(x)^{(s)} = \frac{2\pi}{N}\left(\bar{q}_\mu(x)^{(s)} + \partial_\mu z(x)^{(s)}\right)$$

$$\bar{q}_\mu(x)^{(s)} = \mathrm{sign}(\bar{q}_\mu(x))\left[t^{(s)} \times |\bar{q}_\mu(x)|\right]$$

$$\begin{aligned} z(x)^{(s)} &= \mathrm{sign}(z(x))\left[t^{(s)} \times |z(x)|\right] && \text{(even s)} \\ z(x)^{(s)} &= \begin{cases} \mathrm{sign}(z(x))\left[t^{(s-1)} \times |z(x)|\right] & \text{(even site)} \\ \mathrm{sign}(z(x))\left[t^{(s+1)} \times |z(x)|\right] & \text{(odd site)} \end{cases} && \text{(odd s)} \end{aligned}$$

  – $\tilde{A}_\mu(x)^{(s)}$ is admissible provided that $\boldsymbol{N}$ is large enough

$$|\tilde{F}_{\mu\nu}(x)^{(s)} - t^{(s)}\,\tilde{F}_{\mu\nu}(x)| \leq \frac{8\pi}{N} \quad \ll \epsilon$$

Construction of the measure term

- the measure term as a U(1) lattice gauge field

$$U(x,\mu)^{(+\delta\eta)} = \mathrm{e}^{i\delta\eta_\mu(x)} U(x,\mu) \qquad \mathrm{e}^{i\delta\eta_\mu(x)} \in Z_N(\text{minimal})$$

$$V_\eta \equiv \frac{\det(v_i, v_j^{(+\delta\eta)})}{|\det(v_i, v_j^{(+\delta\eta)})|} \qquad \in U(1)$$

- correspondence

$$\sum_i (v_i, \delta_\eta v_i) \quad \Leftrightarrow \quad \ln V_\eta$$

$$\mathrm{Tr}\left\{\hat{P}_L[\delta_\eta \hat{P}_L, \delta_\zeta \hat{P}_L]\right\} \quad \Leftrightarrow \quad \ln\det\left(1 - \hat{P}_0 + \hat{P}_0 \hat{P}_{+\delta\eta} \hat{P}_{+\delta\eta+\delta\zeta} \hat{P}_{+\delta\zeta} P_0\right)$$

- Gauge anomaly $\quad \mathrm{e}^{i\delta\omega(x)} \in Z_N(\text{minimal})$

$$\exp\left(i\frac{2\pi}{N} q(x)\right) = \det\left(1 - \hat{P} + \hat{P}\mathrm{e}^{i\delta\omega(x)} \prod_i \hat{P}_i \hat{P}\right) \det\left(1 - P_R + P_R \mathrm{e}^{-i\delta\omega(x)}\right)$$
$$\times \prod_{\square\in S} \det\left(1 - \hat{P}_0 + \hat{P}_0 \prod_{i\in\square} \hat{P}_i \hat{P}_0\right)$$

$$q(x) = \partial^*_\mu \bar{k}_\mu(x) \qquad \left(\sum_\alpha e_\alpha^3 = 0\right)$$

- measure term

$$\mathrm{e}^{i\delta\mathfrak{L}_\eta} \equiv \prod_{(\delta\xi,\delta\zeta)\in S} \det\left(1 - \hat{P}_0 + \hat{P}_0 \left\{\prod_{i\in(\delta\xi,\delta\zeta)} \hat{P}_i\right\} \hat{P}_0\right) \mathrm{e}^{i\delta_\eta C}$$

$$\delta_\eta C = \sum_x \tilde{A}_\mu(x)\bar{k}_\mu(x)\Big|_{U^{(+\delta\eta)}} - \sum_x \tilde{A}_\mu(x)\bar{k}_\mu(x)\Big|_U$$

The choice of the measure

$$v_j(x) = \begin{cases} w_j(x)\, \mathrm{e}^{i\phi} & (j=1) \\ w_j(x) & (j \neq 1) \end{cases} \qquad (w_j(x) : \text{arbitrary chosen basis})$$

$$\mathrm{e}^{i\phi} = \det\left(w_i, \hat{P} \prod_{k=0}^{M} \hat{P}^{(k)} \hat{P}^{(0)} w_j^0 \right) \prod_{k=0}^{M} \mathrm{e}^{-i\mathfrak{L}_\eta^{(k)}}$$

Possible Applications

- Still numerically demanding in four dim.
- Ready in two dim.
  - two-dim. chiral Schwinger models
    - Composite massless fermion
    - Check of the cluster property
- Analysis of non-abelian theories ...

$$S = S_B + S_F,$$
$$S_B = \int \mathrm{d}^2x \left\{\partial_\mu \phi^* \partial_\mu \phi + W^{*\prime} W^\prime\right\},$$
$$S_F = \int \mathrm{d}^2x \left\{\bar{\psi}\gamma_\mu \partial_\mu \psi + \bar{\psi} W^{\prime\prime} \frac{1+\gamma_3}{2}\psi + \bar{\psi} W^{*\prime\prime} \frac{1-\gamma_3}{2}\psi\right\}.$$

- Nicolai mapping

$$\phi = \sqrt{\frac{1}{2}}(A+iB), \quad W^\prime = \sqrt{\frac{1}{2}}(U+iV).$$

$$M(x) = -\partial_1 A(x) - \partial_2 B(x) + U(x),$$
$$N(x) = -\partial_2 A(x) + \partial_1 B(x) + V(x),$$

$$\det \begin{pmatrix} \frac{\partial M}{\partial A} & \frac{\partial N}{\partial A} \\ \frac{\partial M}{\partial B} & \frac{\partial N}{\partial B} \end{pmatrix} = \det \left\{\gamma_\mu \partial_\mu + W^{\prime\prime}\frac{1+\gamma_3}{2} + W^{*\prime\prime}\frac{1-\gamma_3}{2}\right\},$$
$$\frac{1}{2}\{M(x)^2 + N(x)^2\} = \partial_\mu \phi^* \partial_\mu \phi + W^{*\prime} W^\prime + W^\prime \partial_{\bar{z}} \phi + W^{*\prime} \partial_z \phi^*$$

- Super-transformation

$$\delta A = \bar{\psi}_1 \xi, \quad \delta B = -i\bar{\psi}_2 \xi$$
$$\delta \psi_1 = -\xi M, \quad \delta \psi_2 = i\xi N$$
$$\delta \bar{\psi}_1 = 0, \quad \delta \bar{\psi}_2 = 0$$

- Nicolai mapping

$$M(x) = (-\nabla_1^S - \nabla_1^A - \nabla_2^A)A(x) - \nabla_2^S B(x) + U(x),$$
$$N(x) = -\nabla_2^S(x) + (\nabla_1^S - \nabla_1^A - \nabla_2^A)B(x) + V(x),$$

$$\nabla_j^S = \frac{1}{2}\left(\nabla_j^+ + \nabla_j^-\right) \ , \ \nabla_j^A = \frac{1}{2}\left(\nabla_j^+ - \nabla_j^-\right).$$

- Jacobian

$$\det \begin{pmatrix} \frac{\partial M}{\partial A} & \frac{\partial N}{\partial A} \\ \frac{\partial M}{\partial B} & \frac{\partial N}{\partial B} \end{pmatrix}$$
$$= \det \left\{ \sum_\mu \left(\gamma_\mu \nabla_\mu^S - \nabla_\mu^A\right) + W'' \frac{1+\gamma_3}{2} + W^{*\prime\prime} \frac{1-\gamma_3}{2} \right\}$$

- Would-be surface term

$$\phi(\nabla_1^S - i\nabla_2^S)W' + \phi^*(\nabla_1^S + i\nabla_2^S)W^{*\prime}$$
$$- \phi(\nabla_1^A + \nabla_2^A)W^{*\prime} - \phi^*(\nabla_1^A + \nabla_2^A)W'$$

– wrong holomorphic structure

Nakayama-Y.K.

$$
\begin{aligned}
S_F &= \sum_x \bar{\psi}(D+F)\psi \\
&= \sum_{x,y} \bar{\psi}(x)\left(D + \frac{1+\gamma_3}{2}W''\frac{1+\hat{\gamma}_3}{2} + \frac{1-\gamma_3}{2}W^{*\prime\prime}\frac{1-\hat{\gamma}_3}{2}\right)_{x,y}\psi(y).
\end{aligned}
$$

$$
D = \begin{pmatrix} T+S_1 & iS_2 \\ -iS_2 & T-S_1 \end{pmatrix},
$$

where T, S_1, S_2 are defined as

$$
T = \frac{1}{a}\left(1 - \frac{1}{\sqrt{X^\dagger X}}\right) - \frac{\nabla_1^A + \nabla_2^A}{\sqrt{X^\dagger X}} = {}^tT,
$$

$$
S_j = \frac{\nabla_j^S}{\sqrt{X^\dagger X}} = -{}^tS_j, \quad j = 1, 2
$$

$$
X = 1 - aD_W.
$$

In this notation, the Ginsparg-Wilson relation can be written as

$$
a(T^2 - S_1^2 - S_2^2) = 2T.
$$

- Nicolai mapping

$$
M = A(T+S_1) + BS_2 + U\left(1 - \frac{a}{2}(T+S_1)\right) - V\frac{a}{2}S_2,
$$

$$
N = AS_2 + B(T-S_1) + V\left(1 - \frac{a}{2}(T-S_1)\right) - U\frac{a}{2}S_2,
$$

- Bosonic action $\quad \Delta = (T^2 - S_1^2 - S_2^2) = 2T/a$

$$
\begin{aligned}
S_B = \sum_x \Big\{ & \phi^*\Delta\phi + W^{*\prime}(1 - \frac{a^2}{4}\Delta)W' \\
& + W'(-S_1 + iS_2)\phi + W^{*\prime}(-S_1 - iS_2)\phi^* \Big\}
\end{aligned}
$$

BRST-like structure in the orbifold construction of SYM

- The action proposed:

$$
\begin{aligned}
S \;=\; & \frac{1}{g^2}\sum_{n}\mathrm{Tr}\Bigg[\frac{1}{2}(\bar{x}_{n-\hat{i}}\,x_{n-\hat{i}} - x_n\,\bar{x}_n + \bar{y}_{n-\hat{j}}\,y_{n-\hat{j}} - y_n\,\bar{y}_n)^2 \\
& +2|x_n y_{n+\hat{i}} - y_n x_{n+\hat{j}}|^2 + d_n d_n \\
& +\sqrt{2}(\alpha_n \bar{x}_n \lambda_n - \alpha_{n-\hat{i}}\lambda_n \bar{x}_{n-\hat{i}}) + \sqrt{2}(\beta_n \bar{y}_n \lambda_n - \beta_{n-\hat{j}}\lambda_n \bar{y}_{n-\hat{j}}) \\
& -\sqrt{2}(\alpha_n y_{n+\hat{i}}\xi_n - \alpha_{n+\hat{j}}\xi_n y_n) + \sqrt{2}(\beta_n x_{n+\hat{j}}\xi_n - \beta_{n+\hat{i}}\xi_n x_n)\Bigg]
\end{aligned}
$$

A mapping:

$$
\begin{aligned}
F^{\lambda}_n &= (\bar{x}_{n-\hat{i}}\,x_{n-\hat{i}} - x_n\,\bar{x}_n + \bar{y}_{n-\hat{j}}\,y_{n-\hat{j}} - y_n\,\bar{y}_n) - i d_n \\
F^{\xi}_n &= 2(x_n y_{n+\hat{i}} - y_n x_{n+\hat{j}})
\end{aligned}
$$

Super-transformation in terms of F^{λ}, F^{ξ}:

$$
\begin{aligned}
\bar{\delta} x_n &= -\sqrt{2} i \alpha_n \\
\bar{\delta} y_n &= -\sqrt{2} i \beta_n \\
\bar{\delta} \lambda_n &= -i(F^{\lambda}_n)^{\dagger} \\
\bar{\delta} \xi_n &= -i(F^{\xi}_n)^{\dagger}
\end{aligned}
$$

$$
\begin{aligned}
\bar{\delta} \bar{x}_n &= 0 \\
\bar{\delta} \bar{y}_n &= 0 \\
\bar{\delta} \alpha_n &= 0 \\
\bar{\delta} \beta_n &= 0
\end{aligned}
$$

$$
\bar{\delta} d_n = \sqrt{2}(\bar{x}_{n-\hat{i}}\,\alpha_{n-\hat{i}} - \alpha_n\,\bar{x}_n + \bar{y}_{n-\hat{j}}\,\beta_{n-\hat{j}} - \beta_n\,\bar{y}_n)
$$

Nilpotency:

$$
(\bar{\delta})^2 = 0
$$

$$
\begin{aligned}
\bar{\delta}(F^{\lambda}_n)^{\dagger} &= 0 \\
\bar{\delta}(F^{\xi}_n)^{\dagger} &= 0
\end{aligned}
$$

BRST-like structure:

$$
\begin{aligned}
\bar{\delta} F_{\rm n}^{\lambda} &= -2\sqrt{2}i(\bar{x}_{\rm n-\hat{i}}\,\alpha_{\rm n-\hat{i}} - \alpha_{\rm n}\,\bar{x}_{\rm n} + \bar{y}_{\rm n-\hat{j}}\,\beta_{\rm n-\hat{j}} - \beta_{\rm n}\,\bar{y}_{\rm n}) \\
\bar{\delta} F_{\rm n}^{\xi} &= -2\sqrt{2}i(\alpha_{\rm n} y_{\rm n+\hat{i}} - y_{\rm n}\alpha_{\rm n+\hat{j}} + x_{\rm n}\beta_{\rm n+\hat{i}} - \beta_{\rm n} x_{\rm n+\hat{j}})
\end{aligned}
$$

$$
\begin{aligned}
S &= \bar{\delta}\left(\sum_{\rm n} \frac{i}{2}\,{\rm Tr}\left[\lambda_{\rm n} F_{\rm n}^{\lambda} + \xi_{\rm n} F_{\rm n}^{\xi}\right]\right) \\
&= \sum_{\rm n} {\rm Tr}\left[\frac{1}{2}|F_{\rm n}^{\lambda}|^2 + \frac{1}{2}|F_{\rm n}^{\xi}|^2 - \frac{i}{2}\lambda_{\rm n}\bar{\delta}F_{\rm n}^{\lambda} - \frac{i}{2}\xi_{\rm n}\bar{\delta}F_{\rm n}^{\xi}\right] \\
&= \sum_{\rm n} {\rm Tr}\Big[\frac{1}{2}(\bar{x}_{\rm n-\hat{i}}\,x_{\rm n-\hat{i}} - x_{\rm n}\,\bar{x}_{\rm n} + \bar{y}_{\rm n-\hat{j}}\,y_{\rm n-\hat{j}} + y_{\rm n}\,\bar{y}_{\rm n})^2 + \frac{1}{2} d_{\rm n} d_{\rm n} \\
&\quad + 2|x_{\rm n} y_{\rm n+\hat{i}} - y_{\rm n} x_{\rm n+\hat{j}}|^2 \\
&\quad + \sqrt{2}(\alpha_{\rm n}\bar{x}_{\rm n}\lambda_{\rm n} - \alpha_{\rm n-\hat{i}}\lambda_{\rm n}\bar{x}_{\rm n-\hat{i}}) + \sqrt{2}(\beta_{\rm n}\bar{y}_{\rm n}\lambda_{\rm n} - \beta_{\rm n-\hat{j}}\lambda_{\rm n}\bar{y}_{\rm n-\hat{j}}) \\
&\quad - \sqrt{2}(\alpha_{\rm n} y_{\rm n+\hat{i}}\xi_{\rm n} - \alpha_{\rm n+\hat{j}}\xi_{\rm n} y_{\rm n}) + \sqrt{2}(\beta_{\rm n} x_{\rm n+\hat{j}}\xi_{\rm n} - \beta_{\rm n+\hat{i}}\xi_{\rm n} x_{\rm n}) \quad \Big]
\end{aligned}
$$

Fermionic part of the action:

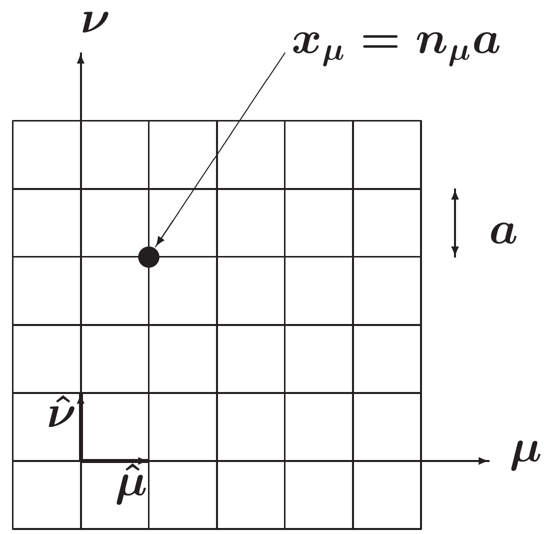
$$
\begin{aligned}
S_F &= \sum_{\rm n} {\rm Tr}\Big[+\sqrt{2}(\alpha_{\rm n}\bar{x}_{\rm n}\lambda_{\rm n} - \alpha_{\rm n-\hat{i}}\lambda_{\rm n}\bar{x}_{\rm n-\hat{i}}) \\
&\qquad + \sqrt{2}(\beta_{\rm n}\bar{y}_{\rm n}\lambda_{\rm n} - \beta_{\rm n-\hat{j}}\lambda_{\rm n}\bar{y}_{\rm n-\hat{j}}) \\
&\qquad - \sqrt{2}(\alpha_{\rm n} y_{\rm n+\hat{i}}\xi_{\rm n} - \alpha_{\rm n+\hat{j}}\xi_{\rm n} y_{\rm n}) \\
&\qquad + \sqrt{2}(\beta_{\rm n} x_{\rm n+\hat{j}}\xi_{\rm n} - \beta_{\rm n+\hat{i}}\xi_{\rm n} x_{\rm n}) \quad \Big] \\
&= -\sum_{\rm n,m} {\rm Tr}\sqrt{2}\begin{pmatrix} \alpha_{\rm m} & \beta_{\rm m} \end{pmatrix} \begin{pmatrix} \frac{\delta}{\delta x_{\rm m}} F_{\rm n}^{\lambda} & \frac{\delta}{\delta x_{\rm m}} F_{\rm n}^{\xi} \\ \frac{\delta}{\delta y_{\rm m}} F_{\rm n}^{\lambda} & \frac{\delta}{\delta y_{\rm m}} F_{\rm n}^{\xi} \end{pmatrix} \begin{pmatrix} \lambda_{\rm n} \\ \xi_{\rm n}/2 \end{pmatrix}
\end{aligned}
$$

Lattice Gauge Theory

$$x_\mu = n_\mu a$$

$$\Longleftrightarrow$$

$$p_\mu \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$$



The fermion field is introduced on the lattice site

$$\psi(x) \qquad x_\mu = n_\mu a \ (n_\mu \in Z)$$

The differential of the field can be replaced by difference:

$$\partial_\mu \psi(x) = \frac{1}{a}\left(\delta_{x+\hat{\mu},y} - \delta_{x,y}\right)\psi(y) = \frac{1}{a}\left(\psi(x+\hat{\mu}) - \psi(x)\right)$$

Gauge-covariance on the lattice

- Link variable and its gauge transformation:

$$U_\mu(x) = \mathrm{e}^{iaA_\mu(x)} \in G, \quad U_\mu(x) \to g(x)U_\mu(x)g^{-1}(x+\hat{\mu})$$

- Gauge-covariant difference operator:

$$\nabla_\mu \psi(x) = \frac{1}{a}\left(U_\mu(x)\psi(x+\hat{\mu}) - \psi(x)\right)$$

- Field strength of lattice gauge field

$$[\nabla_\nu, \nabla_\mu]\,\psi(x) \ = \ (1 - P_{\mu\nu}(x))\ U_\nu(x)U_\mu(x+\hat{\nu})\psi(x+\hat{\mu}+\hat{\nu})$$

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\nu(x+\hat{\nu})^{-1}U_\nu(x)^{-1}$$

Species Doubling and Nielsen-Ninomiya Theorem

Fermion action discretized on the lattice:

$$\begin{aligned} S &= a^4 \sum_x \bar{\psi}(x)\,\gamma_\mu \frac{1}{2}\left(\partial_\mu - \partial_\mu^\dagger\right)\,\psi(x) \\ &= \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4}\,\bar{\psi}(-k)\,\left\{ i\gamma_\mu \frac{1}{a}\sin k_\mu a \right\}\,\psi(k) \end{aligned}$$

Propagator of the fermion:

$$\langle \psi(k)\,\bar{\psi}(-k)\rangle = \frac{-i\gamma_\mu \frac{1}{a}\sin k_\mu a}{\sum_\nu \frac{1}{a^2}\sin^2 k_\nu a}$$

Poles appear at $k_\mu a = (0,0,0,0), (\pi,0,0,0), \cdots, (\pi,\pi,\pi,\pi)$

$\Longrightarrow$ Inevitable due to Nielsen-Ninomiya Theorem

$$S = \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4}\,\bar{\psi}(-k)\,\tilde{D}(k)\,\psi(k)$$

1. $\tilde{D}(k)$ is a periodic and analytic function of momentum k_μ
2. $\tilde{D}(k) \propto i\gamma_\mu k_\mu$ for $|k_\mu| a \ll \pi$
3. $\tilde{D}(k)$ is invertible for all k_μ except $k_\mu = 0$
4. $\gamma_5 \tilde{D}(k) + \tilde{D}(k)\gamma_5 = 0$

These four conditions cannot be satisfied simultaneously.

note: analyticity and locality

$$\begin{aligned} &\frac{\partial^l}{\partial k^l}\tilde{D}(k) = \sum_x e^{ikx}(ix)^l D(x) < \infty \\ &\Longrightarrow \qquad \|D(x)\| < Ce^{-\gamma|x|} \end{aligned}$$

Wilson term and Chiral Symmetry Breaking

Lift the degeneracy due to the species doublers
by the mass term which consists of a second-order derivative operator:

$$S_{\mathrm{w}} = a^4 \sum_x \bar{\psi}(x) \left(\gamma_\mu \frac{1}{2} \left(\partial_\mu - \partial_\mu^\dagger \right) + \frac{a}{2} \left(\partial_\mu \partial_\mu^\dagger \right) + m \right) \psi(x)$$

The Wilson mass term in the momentum space:

$$\sum_\mu \frac{a}{2} \left(\frac{2}{a} \sin \frac{k_\mu a}{2} \right)^2 \simeq \frac{2n}{a}$$

where n is the number of π of the "zero momentum" for the species doublers
All the species doublers, except the physical one, get masses of the order of the inverse lattice spacing and decouple in the continuum limit, $a \longrightarrow 0$

Lost of manifest chiral symmetry:

- $SU(N_f)_L \times SU(N_f)_R$ flavor chiral symmetry
- axial $U(1)$ anomaly due to the explicit breaking

Renormalization Group Analysis of Chiral Symmetry Breaking

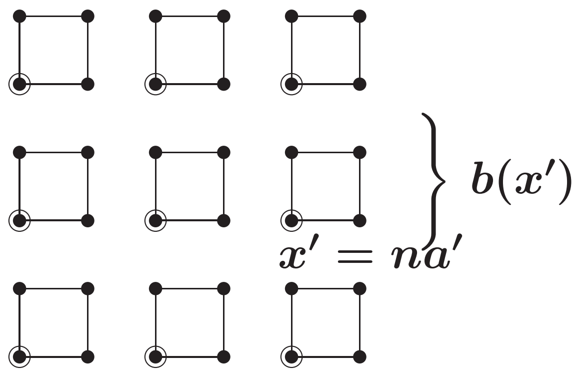
Chiral Symmetry Breaking due to Heavy Species Doublers
Single massless Dirac fermion for sufficiently small momentum

$$|k_\mu| \ll \pi/a$$

Block Spin Transformation Ginsparg and Wilson

$\Longrightarrow$ local low energy effective action for the massless mode ?!

$$\psi'(x') = Z \frac{1}{2^4} \sum_{x \in b(x')} \psi(x) = B(x';\psi)$$



Effective action for the blocked variables:

$$e^{-S'[\psi',\bar{\psi}']} = \int \prod_x d\psi(x) d\bar{\psi}(x)\, e^{-S_W[\psi,\bar{\psi}]} \times \exp\left\{ -\alpha \sum_{x'} \left(\bar{\psi}'(x') - B(x';\bar{\psi})\right)\left(\psi'(x') - B(x';\psi)\right) \right\}$$

Fixed point D^*:

$$S^* = a^4 \sum_x \bar{\psi}(x) D^* \psi(x)$$

It turns out that this effective Dirac operator satisfies the relation:

$$\gamma_5 D^* + D^* \gamma_5 = a D^* \gamma_5 R D^*, \qquad R = \frac{2}{\alpha}$$

$$
\begin{aligned}
e^{-S'[\psi',\bar\psi']} &= \int \prod_x d\psi(x) d\bar\psi(x) \prod_{x'} d\eta(x') d\bar\eta(x') \times \\
&\quad \exp\Big\{ -\sum_x \bar\psi(x) D_{\rm w}(x,y)\psi(y) - \frac{1}{\alpha}\sum_{x'} \bar\eta(x')\eta(x') \times \\
&\quad\quad + \sum_{x'} \left(\bar\psi'(x') - B(x';\bar\psi)\right)\eta(x') + \sum_{x'} \bar\eta(x')\left(\psi'(x') - B(x';\psi)\right)\Big\} \\
&= \int \prod_{x'} d\eta(x') d\bar\eta(x') \exp\Big\{ \sum_{x'} \left(\bar\psi'(x')\eta(x') + \bar\eta(x')\psi'(x')\right)\Big\} \times \\
&\quad \exp\Big\{ -\sum_{x'} \bar\eta(x') \Big(\left(\tfrac{b}{2^d}\right)^2 \sum_{x\in b(x'), y\in b(y')} D_{\rm w}(x,y)^{-1} + \frac{1}{\alpha}\delta(x',y')\Big)\eta(x')\Big\}
\end{aligned}
$$

$$
D'(x',y')^{-1} = \left(\frac{b}{2^d}\right)^2 \sum_{x\in b(x'), y\in b(y')} D_{\rm w}(x,y)^{-1} + \frac{1}{\alpha}\delta(x',y')
$$

Momentum space:

$$
k_\mu = \frac{1}{2}\left(k'_\mu + 2\pi l_\mu\right), \qquad k_\mu, k'_\mu \in [-\pi,\pi], \quad l_\mu = 0,1
$$

$$
\begin{aligned}
D'(k)^{-1} &= -i\gamma_\mu \alpha'_\mu(k) + \beta'(k) \\
\alpha'_\mu(k) &= \sum_{l_\nu=0,1} \left(\frac{b^2}{2^d}\right) \prod_\nu \left(\frac{\sin\frac{k_\nu}{2}}{2\sin\frac{1}{2}\left(\frac{k_\nu+2\pi l_\nu}{2}\right)}\right)^2 \alpha_\mu^{(\rm w)}\left(\frac{k_\nu+2\pi l_\nu}{2}\right) \\
\beta'(k) &= \sum_{l_\nu=0,1} \left(\frac{b^2}{2^d}\right) \prod_\nu \left(\frac{\sin\frac{k_\nu}{2}}{2\sin\frac{1}{2}\left(\frac{k_\nu+2\pi l_\nu}{2}\right)}\right)^2 \beta^{(\rm w)}\left(\frac{k_\nu+2\pi l_\nu}{2}\right) + \frac{1}{\alpha}
\end{aligned}
$$

Fixed point solution:

$$
\left(\frac{b^2}{2^d}\right) 2 = 1
$$

$$
\alpha^*_\mu(k) = \sum_{l_\nu\in Z} \prod_\nu \left(\frac{\sin\frac{k_\nu}{2}}{2\left(\frac{k_\nu+2\pi l_\nu}{2}\right)}\right)^2 \frac{k_\mu + 2\pi l_\mu}{(k+2\pi l)^2}
$$

$$
\beta^*(k) = \frac{2}{\alpha}
$$

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

In terms of the fermion propagator $S_F = D^{-1}$:

$$\gamma_5 S_F(x,y) + S_F(x,y)\gamma_5 = \gamma_5 2\delta(x,y)$$

- chiral symmetry is broken only in local contact terms
- such local terms do not contribute to the physical amplitudes evaluated at long-distance, $x - y \neq 0$
- represents the chiral limit for lattice Dirac operators in a consistent manner with Nielsen-Ninomiya theorem

Exact Chiral Symmetry on the Lattice

Ginsparg-Wilson relation implies an exact symmetry of the fermion action ! Lüscher

Under the following transformation,

$$\delta\psi(x) = \gamma_5 (1 - 2aD)\,\psi(x), \quad \delta\bar{\psi}(x) = \bar{\psi}(x)\gamma_5$$

the fermion action is invariant,

$$\delta S = a^4 \sum_x \bar{\psi}\,\{D\gamma_5 (1 - 2aD) + \gamma_5 D\}\,\psi(x) = 0$$

- local transformation as long as D is local
- can be regarded as the lattice counterpart of chiral symmetry in the continuum theory

Neuberger's lattice Dirac operator

Gauge-covariant form : Neuberger

$$D = \frac{1}{2a}\left(1 + X\frac{1}{\sqrt{X^\dagger X}}\right) = \frac{1}{2a}\left(1 + \gamma_5\frac{H}{\sqrt{H^2}}\right)$$

$$X = \left(D_{\rm w} - \frac{m_0}{a}\right), \quad H = \gamma_5 X, \quad (0 < m_0 < 2)$$

$$D_{\rm w} = \sum_\mu \left\{\gamma_\mu \frac{1}{2}\left(\nabla_\mu - \nabla_\mu^\dagger\right) + \frac{a}{2}\left(\nabla_\mu \nabla_\mu^\dagger\right)\right\}$$

free fermion :

$$\tilde{D}(p) = \frac{1}{2}\left(1 + \frac{i\gamma_\mu \bar{p}_\mu + \frac{1}{2}\hat{p}^2 - m_0}{\sqrt{\bar{p}^2 + \left(\frac{1}{2}\hat{p}^2 - m_0\right)^2}}\right)$$

$$\bar{p}_\mu = \sin p_\mu, \quad \hat{p}_\mu = 2\sin\frac{p_\mu}{2}$$

- analytic periodic function in momentum p_μ for $m_0 \in (0, 2)$

indeed local !

- $\tilde{D}(p) \simeq Z\, i\gamma_\mu p_\mu \quad (|p| \ll \pi\ )$
- $\tilde{D}(p) \simeq 1 \quad (|p| \simeq \pi\ )$
- $\gamma_5 \tilde{D}(p)^{-1} + \tilde{D}(p)^{-1}\gamma_5 = 2\gamma_5$

Chiral symmetry breaking only in local contact term !

Consistent with Nielsen-Ninomiya theorem !

1. Locality   Hernándes, Jansen and Lüscher

Non-trivial due to inverse square root of the hermitian Wilson-Dirac operator

$$\frac{H}{\sqrt{H^2}}$$

Eigenvalue spectrum of H^2 is closely related to the size of field strength of lattice gauge field

$$\|H^2\| \geq \left\{(1-30\epsilon)^{\frac{1}{2}} - |1-m_0|\right\}^2 > 0$$

for

$$\| 1 - P_{\mu\nu}(x) \| < \epsilon, \qquad \epsilon < \frac{1}{30}\left(1-|1-m_0|^2\right)$$

$(m_0 = 1)$

$$\begin{aligned} X &= \sum_\mu \left\{\gamma_\mu \frac{1}{2}\left(\nabla_\mu + \nabla_\mu^*\right) - \frac{a}{2}\nabla_\mu\nabla_\mu^*\right\} - \frac{1}{a} \\ &= \sum_\mu \left\{\gamma_\mu \frac{1}{2}\left(\nabla_\mu + \nabla_\mu^*\right) - \frac{1}{2}\left(\nabla_\mu - \nabla_\mu^*\right)\right\} - \frac{1}{a} \end{aligned}$$

$$a^2 X^\dagger X = 1 + \frac{1}{4}\sum_{\mu\neq\nu}\{B_{\mu\nu} + C_{\mu\nu} + D_{\mu\nu}\}$$

$$\begin{aligned} B_{\mu\nu} &= a^4 \nabla_\mu^*\nabla_\mu\nabla_\nu^*\nabla_\nu = a^4\nabla_\mu^*\nabla_\nu^*\nabla_\nu\nabla_\mu - a^3\nabla_\mu^*\left[\nabla_\mu, \nabla_\nu^* - \nabla_\nu\right] \\ C_{\mu\nu} &= \frac{1}{2} i\sigma_{\mu\nu} a^2 \left[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* + \nabla_\nu\right] \\ D_{\mu\nu} &= -\gamma_\mu a^2\left[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* - \nabla_\nu\right] \end{aligned}$$

$$\| a^2\left[\nabla_\mu, \nabla_\nu\right] \| < \epsilon$$

$$a^2 X^\dagger X \geq 1 - 30\epsilon$$

$$\left\| \frac{1}{\sqrt{H^2}}(x, y) \right\| < \frac{\kappa}{1-t} \exp\{-\theta|x-y|/2a\}$$

- Assume H^2 is bounded from below and above by positive constant

$$0 < \alpha < \| a^2 H^2 \| < \beta$$

- Consider the generating function of the Legendre polynomials

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{k=0}^{\infty} t^k P_k(z), \qquad \|P_k(z)\| \leq 1$$

- Set

$$z = \frac{\beta + \alpha - 2H^2}{\beta - \alpha}$$

$$\cosh\theta = \frac{\beta-\alpha}{\beta+\alpha}, \quad t = e^{-\theta}, \quad \kappa = \sqrt{\frac{4t}{\beta-\alpha}}$$

$$\frac{1}{\sqrt{H^2}} = \kappa \sum_{k=0}^{\infty} t^k P_k(z)$$

2. Index theorem on the lattice

Chiral transformation depending on gauge fields

$$\delta\psi(x) = \gamma_5 (1 - 2aD)\, \psi(x), \quad \delta\bar{\psi}(x) = \bar{\psi}(x)\gamma_5$$

Chiral Jacobian

$$\delta\left[\prod_x d\psi(x)\, d\bar{\psi}(x)\right] = \left[\prod_x d\psi(x)\, d\bar{\psi}(x)\right] (-2)\mathrm{Tr}\gamma_5(1 - aD)$$

- Chiral anomaly in the (classical) continuum limit   A. Yamada and Y.K.

  K. Fujikawa, H. Suzuki, D. Adams

$$-\frac{2}{a^4}\mathrm{tr}\gamma_5(1 - aD)(x, x) = \frac{g^2}{32\pi^2}\epsilon_{\mu\nu\lambda\rho}F^a_{\mu\nu}(x)F^a_{\lambda\rho}(x) + \mathcal{O}(a)$$

- Index theorem at a finite lattice spacing   P. Hasenfratz et al.

  Zero modes of D are chiral eigenstates !

$$\begin{aligned} D\psi_0(x) &= 0 \\ D\gamma_5\psi_0(x) &= (-\gamma_5 D + 2aD\gamma_5 D)\,\psi_0 = 0 \end{aligned}$$

  Index theorem

$$2N_f\, \mathrm{Index}(D) = -2\mathrm{Tr}\gamma_5(1 - aD)$$

$$\begin{aligned} &-2a(D+m)\gamma_5(D+m) \\ &= 2m(1+am)\gamma_5 - (1+2am)\left\{(D+m)\gamma_5 + \gamma_5(D+m)\right\} \end{aligned}$$

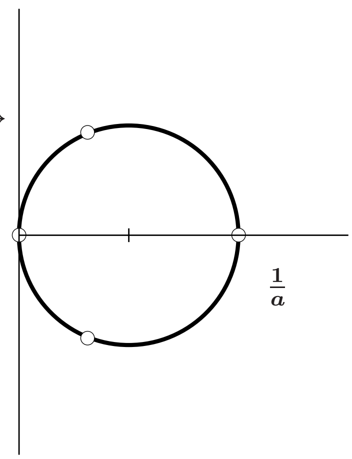
$$-2a\mathrm{Tr}\gamma_5 D = 2m(1+am)\mathrm{Tr}\gamma_5\frac{1}{D+m}$$

- Eigenvalues of D

$$D + D^\dagger = 2aD^\dagger D = 2aDD^\dagger \quad \text{(normal)}$$
$$D^\dagger = \gamma_5 D \gamma_5 \quad (\gamma_5\text{-conjugate})$$

$$\begin{aligned} \lambda + \lambda^* - 2a\lambda^*\lambda &= (-2a)\left[(\lambda - 1/2a)(\lambda - 1/2a)^* - (1/2a)^2\right] \\ &= 0 \end{aligned}$$

$\therefore$ on the circle with center $(1/2a, 0)$ and radius $1/2a$



$$\begin{array}{lcll} \lambda = 0 & : & \gamma_5\psi_\lambda(x) = \pm\psi_\lambda(x) & n_\pm \\ \lambda = 1/a & : & \gamma_5\psi_\lambda(x) = \pm\psi_\lambda(x) & N_\pm \\ \lambda \neq 0, 1/a & : & \text{pair-wise} \left\{ \begin{array}{l} \lambda \rightarrow \psi_\lambda \\ \lambda^* \rightarrow \gamma_5\psi_\lambda \end{array} \right. & \psi_\lambda^\dagger \gamma_5 \psi_\lambda = 0 \end{array}$$

- Index theorem

$$\begin{aligned} \mathrm{Tr}\,\{\gamma_5(1 - aD)\} &= \sum_\lambda \psi_\lambda^\dagger\gamma_5\psi_\lambda - a\sum_\lambda \lambda\,\psi_\lambda^\dagger\gamma_5\psi_\lambda \\ &= \sum_{\lambda=0,1/a} \psi_\lambda^\dagger\gamma_5\psi_\lambda - a\sum_{\lambda=1/a} \frac{1}{a}\,\psi_\lambda^\dagger\gamma_5\psi_\lambda \\ &= (n_+ - n_-) + (N_+ - N_-) - (N_+ - N_-) \\ &= n_+ - n_- \end{aligned}$$

Topological charge of lattice gauge fields $\{U_\mu(x)\}$

$$Q = \sum_x \mathrm{tr}\,\{\gamma_5(1-aD)(x,x)\} = -\frac{1}{2}\sum_x \mathrm{tr}\left\{\frac{H}{\sqrt{H^2}}(x,x)\right\}$$

- Fermionic definition (using Wilson-Dirac operator)

  Narayanan-Neuberger(1995), Ito-Iwasaki-Yoshie(1987)

- depending on $\{U_\mu(x)\}$ smoothly and locally

Lattice gauge fields $\{U_\mu(x)\}$ does not have any topological structure
Any two lattice gauge fields are smoothly connected ?!

A sufficient condition for non-trivial topological structure of $\{U_\mu(x)\}$

$$\|1 - P_{\mu\nu}(x)\| < \epsilon$$

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu(x+\hat{\nu})^{-1}U_\nu(x)^{-1}$$

cf. Geometrical definition Lüscher (1982), Pillips-Stone (1996)
cf. Wilson action Continuum limit (Weak coupling limit)

$$S_G = \frac{1}{g^2}\sum_x\sum_{\mu\nu}\mathrm{ReTr}\,(1 - P_{\mu\nu}(x))$$

Locality of D	$\Rightarrow$	GW rel. $\gamma_5 D + D\gamma_5 = 2D\gamma_5 D$	$\Rightarrow$	Index of D
		$\searrow$		$\Downarrow$
$\Uparrow$		Local chiral anomaly		Index theorem on the lattice
		$\sum_x -\mathrm{tr}\gamma_5 D(x,x)$ $\searrow$		$\Uparrow$
Admissibility cond. $\|1 - P_{\mu\nu}(x)\| < \epsilon$	$\Rightarrow$	Topological structure of lattice gauge fields	$\Rightarrow$	Topological charge $Q = -\frac{1}{2}\sum_x \mathrm{tr}\gamma_5 D(x,x)$

5. Unitarity

Free Overlap Dirac fermion Lüscher

Spectral representation of free fermion propagator

$$aD = 1 - A(A^\dagger A)^{-1/2},\, A = 1 - aD_{\rm w}$$
$$D_{\rm w} = \frac{1}{2}\left\{\gamma_\mu\left(\partial_\mu^* + \partial_\mu\right) - a\partial_\mu^*\partial_\mu\right\}$$

$$\langle\psi(x)\bar\psi(y)\rangle_{x_0>y_0} = \int_0^\infty dE \int_{-\pi/a}^{\pi/a} \frac{d^3\mathrm{p}}{(2\pi)^3}\, \rho(E,\mathrm{p})\, \mathrm{e}^{-E(x_0-y_0)+i\mathrm{p}(\mathrm{x}-\mathrm{y})}$$

such that

$$dEd^3\mathrm{p}\, \zeta^\dagger\, \rho(E,\mathrm{p})\, \zeta \ge 0$$

for all complex Dirac spinors.

$$\begin{aligned}\rho(E,p) &= (\gamma_0 \sinh E - i\gamma_k \sin p_k) \\ &\times \left\{\delta(E-\omega_p)\theta\left(\cosh E - \frac{1}{2}\hat p^2\right)\frac{\cosh E - \frac{1}{2}\hat p^2}{\sinh 2E}\right. \\ &\left. + \frac{1}{2\pi}\theta(E-E_p)\frac{\left\{\hat p^2\left(\cosh E - \cosh E_p\right)\right\}^{1/2}}{\hat p^2\left(\cosh E - \cosh E_p\right) + \left(\cosh E - \frac{1}{2}\hat p^2\right)^2}\right\}\end{aligned}$$

Unitarity is OK for any value of a!

4. Universality class

OK with the admissibility condition
How about with Wilson action ?

$$S_G = \frac{1}{g^2} \sum_x \sum_{\mu\nu} \mathrm{ReTr}\,(1 - P_{\mu\nu}(x))$$

Small and zero eigenvalues of H

1. Isolated (almost) zero modes

   local large fluctuation of field strength $1 - P_{\mu\nu}(x)$

   $\frac{H}{\sqrt{H^2}}$ remains local $\Leftarrow$ localized zero modes $\phi_0(x)$ Hernándes-Jansen-Lüscher

   $\sharp$(localized zero mode) $\propto L^4$ Kiskis-Narayanan-Neuberger

2. Collapse of Continuum spectrum

   global large fluctuation of field strength $1 - P_{\mu\nu}(x)$

cf. Aoki (Parity-Flavor broken) phase of Wilson-Dirac fermion S. Aoki
order parameter (two-flavor Wilson fermion):

$$\lim_{h\to 0} \langle \bar\psi(x)\, i\gamma_5 \sigma_3\, \psi(x)\rangle_h = \lim_{h\to 0} \left\langle \mathrm{Tr} \frac{h}{\left(\gamma_5 \left(D_{\rm w} + \frac{m_0}{a}\right)\right)^2 + h^2} \right\rangle_h$$

Nonzero density of zero eigenvalues of the Wilson-Dirac operator indeed triggers the phase transition

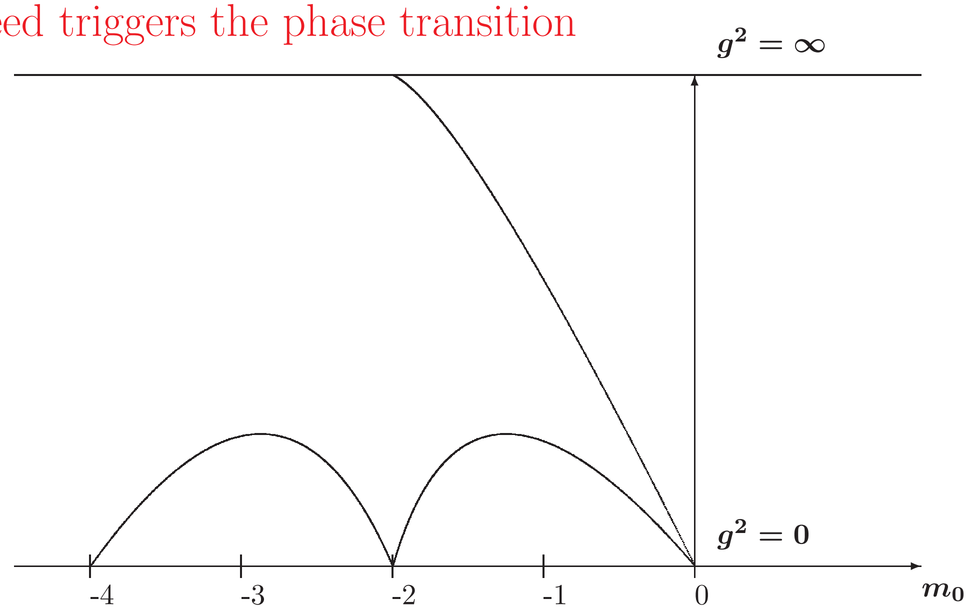
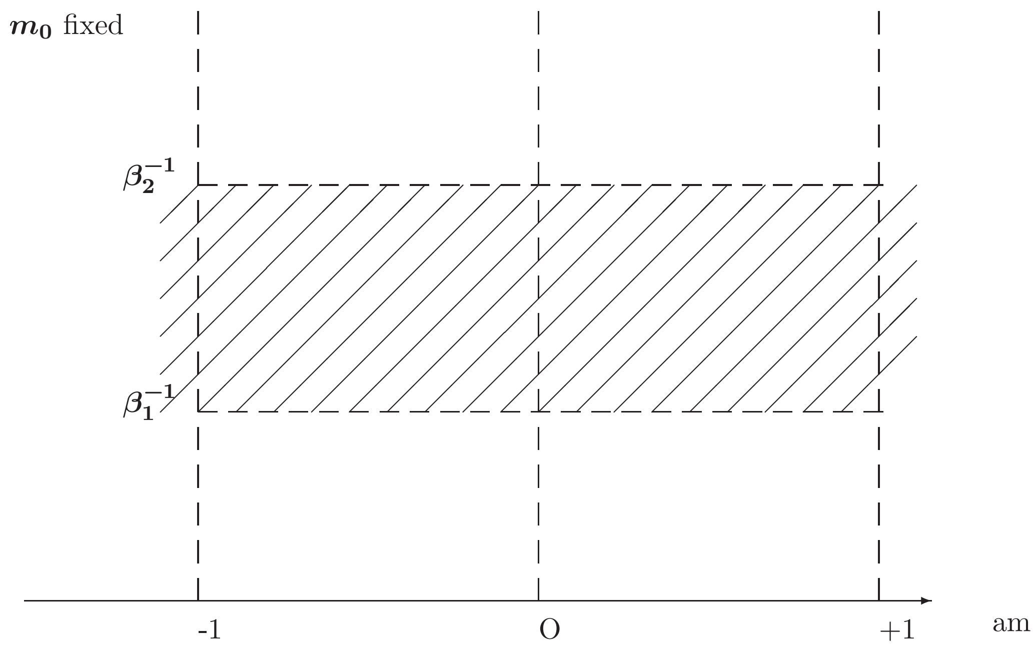


Figure 1: Phase Diagram of Wilson Fermion

cf. Strong coupling region Brower et al, Ichinose-Nagao, Golterman-Shamir



- $\beta > \beta_1$
  Smooth and Local, $\hat{\gamma}_5^2 = 1$
- $\beta_1 > \beta > \beta_2$
  Not local and smooth
- $\beta < \beta_2$
  Smooth and Local, $\hat{\gamma}_5^2 = 1$, but no massless fermion

$$m_0 \simeq 1.0 \quad \beta_1 > 5.7$$
$$m_0 \simeq 1.6 \quad \beta_1 < 5.85 \quad 1/a \simeq 1.5 GeV$$

Hernández-Jansen-Lellouch (1999)

Practical implementations of Neuberger's Dirac operator

How to implement the sign function:

$$\epsilon(H) = \frac{H}{\sqrt{H^2}}$$

1. Legendre polynomial expansion Bunk, Hernándes-Jansen-Lüscher

$$\frac{1}{\sqrt{H^2}} = \kappa \sum_{k=0}^{\infty} t^k P_k(z) \qquad z = \frac{\beta + \alpha - 2H^2}{\beta - \alpha}$$

2. Rational approximation and Domain wall fermion Neuberger, Boriçi

$$\epsilon_N(H) = \frac{(1+H)^N - (1-H)^N}{(1+H)^N + (1-H)^N}$$

(a)

$$H = \gamma_5 (D_w - \frac{m_0}{a})$$

$$\epsilon_N(H) = \frac{H}{n} \sum_{s=1}^{n} \frac{1}{\cos^2\left(\frac{\pi(s-1/2)}{2n}\right)} \frac{1}{\tan^2\left(\frac{\pi(s-1/2)}{2n}\right) + H^2}$$

(b)

$$H = \gamma_5 (D_w - \frac{m_0}{a}) \frac{1}{2 + a_5(D_w - \frac{m_0}{a})}$$

Transfer matrix of 4+1 dim. Wilson-Dirac fermion

$$\frac{1-H}{1+H} = T$$

$$aD_N = \frac{1}{2}\left(1 + \gamma_5 \epsilon_N(H)\right) = \left(P_R + P_L T^N\right)/(1 + T^N)$$