From Liouville to SL(2,R) WZNW model

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"Zamolodchikov relations and Liouville hierarchy in SL(2,R)(k) WZNW model'' hep-th/0409227 published in NPB

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"The Stoyanovsky-Ribault-Teschner map and string scattering amplitudes'' hep-th/0505203

Introduction

- The intimate relation between Liouville theory and SL(2,R) WZNW model dates back to early days of quantum gravity.

- Early hope: SL(2,R) is more tractable?
- In reality: Liouville is better understood.

- Application of LFT:
  - Noncritical string (matrix model)
  - 2D quantum gravity
  - Homogeneous tachyon condensation
- Application of SL(2,R) WZNW model
  - AdS/CFT
  - Little String Theory (2D BH).
  - Nonhomogeneous tachyon condensation

- Zamolodchikov's equation in Liouville and SL(2,R) WZNW model.

$$D_m \bar{D}_m [\varphi e^{\frac{1-m}{2}\varphi}] = B_m e^{\frac{1+m}{2}\varphi}$$

$$\partial_x^m \partial_{\bar{x}}^m \tilde{\Phi}_m = -m!(m-1)!\Phi_{-m}$$

- Stoyanovsky-Ribault-Teschner map.

$$\langle \Phi_{j_1,m_1,\bar{m}_1} \cdots \Phi_{j_N,m_N,\bar{m}_N} \rangle^{SL(2,R)}$$

$$\sim \langle \prod_{t=1}^{2N-2} V_{\alpha_t}(z_t) \rangle^{Liouville}$$

Liouville Field Theory

- Action

$$S = \frac{1}{2\pi}\int d^2z \left(\partial\varphi\bar{\partial}\varphi + \frac{\sqrt{2}}{4}QR\varphi + 2\pi\mu e^{\sqrt{2}b\varphi}\right)$$

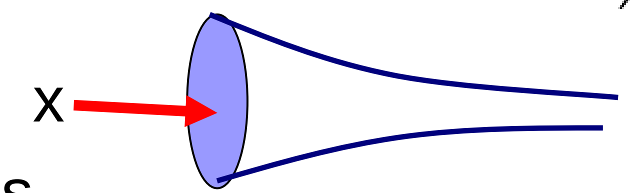
- Central charge: $c = 1 + 6Q^2, Q = b + b^{-1}$
- Vertex operator: $V_\alpha \sim e^{\sqrt{2}\alpha\varphi}$
- Dimension: $\Delta_\alpha = \alpha(Q - \alpha)$
- Structure constants are known (DOZZ, Teschner)
- Classical limit is $b \to \infty, \quad 2b\varphi \to \varphi^c$

SL(2,R) WZNW model

- Euclideanized action (classical limit $k \to \infty$)

$$S = \frac{1}{2\pi}\int d^2z \left(\partial\phi\bar{\partial}\phi - \frac{\sqrt{2}}{4}\frac{1}{\sqrt{k-2}}R\phi + \partial\bar{\gamma}\bar{\partial}\gamma e^{-\sqrt{\frac{2}{k-2}}\phi}\right)$$

- Central charge c = 3+6/(k-2).



- Vertex operator in x (harmonic) basis

$$\Phi_j(x) = \frac{2j+1}{\pi}\left(|\gamma - x|^2 e^{\phi} + e^{-\phi}\right)^{2j}$$

$$\Delta_j = -\frac{j(j+1)}{k-2}$$

- m (Cartan-eigenvalue) basis $J_3|j,m\rangle = m|j,m\rangle$

$$\Phi_{j,m,\bar{m}} = \int d^2x \Phi_j(x) x^{-1-j+m}\bar{x}^{-1-j+\bar{m}}$$

Zamolodchikov's higher equation of motion and USO

Al. Zamolodchikov: hep-th/0312279

G. Bertoldi, G. Giribet: hep-th/0405094

Zamolodchikov relation for Liouville theory

- Liouville Field Theory contains two characteristic equations
- Decoupling equation (Virasoro Null Vector)

$$D_1 \cdot 1 = \partial_z \cdot 1 = 0 \ ,$$

$$D_2 \cdot e^{-\varphi^c/2} = (\partial_z^2 + \frac{1}{2}T)e^{-\varphi^c/2} = 0 \ ,$$

$$D_3 \cdot e^{-\varphi^c} = (\partial_z^3 + 2T\partial_z + T')e^{-\varphi^c} = 0 \ ,$$

$$D_4 \cdot e^{-3\varphi^c/2} = (\partial_z^4 + 5T\partial_z^2 + 5T'\partial_z + (\frac{9}{4}T^2 + \frac{3}{2}T''))e^{-3\varphi^c/2} = 0 \ .$$

- One to one corresponding higher equation of motion

$$D_m \bar{D}_m [\varphi^c e^{\frac{1-m}{2}\varphi^c}] = B_m e^{\frac{1+m}{2}\varphi^c}$$

$$B_m = (-2)^{1-m} (\mu^c)^m m!(m-1)!$$

- Quantum mechanically, they hold as operator valued equations.
- Important for solvability of noncritical string (2D gravity)?

General argument

- Consider Virasoro decoupling operator $\bar{D}_{m,n}$

$$\bar{D}_{m,n} V_\alpha \sim (\alpha - \alpha_{m,n}) A_{m,n} + O((\alpha - \alpha_{m,n})^2)$$

$$\alpha_{m,n} = -(m-1)b^{-1}/2 - (n-1)b/2$$

- $A_{m,n}$ is no more right primary but still left primary.

- So $D_{m,n} A_{m,n} = D_{m,n} \bar{D}_{m,n} V'_{\alpha_{m,n}}$ $\qquad V'_\alpha = \frac{1}{2}\frac{\partial}{\partial\alpha} V_\alpha$

  is a primary field (note V' is logarithmic primary).

- Because we assume all primaries are spanned by V,

$$D_{m,n} \bar{D}_{m,n} V'_{\alpha_{m,n}} = B_{m,n} V_{\tilde{\alpha}_{m,n}}$$

$$\tilde{\alpha}_{m,n} = -(m-1)b^{-1}/2 + (n+1)b/2$$

Zamolodchikov relation for SL(2,R) WZNW

- Similar construction is possible for any (solvable) irrational CFT.

$$\partial_{\bar{x}}^m \Phi_m = \partial_x^m \Phi_m = 0$$

$$\partial_x^m \partial_{\bar{x}}^m \tilde{\Phi}_m = -m!(m-1)!\Phi_{-m}$$

$$\Phi_m(z|x) = \frac{m}{\pi}(|x-\gamma(z)|^2 e^{\phi(z)} + e^{-\phi(z)})^{m-1}$$

$$\tilde{\Phi}_m(z|x) = \frac{m}{\pi}(|x-\gamma(z)|^2 e^{\phi(z)} + e^{-\phi(z)})^{m-1} \ln(|x-\gamma(z)|^2 e^{\phi(z)} + e^{-\phi(z)})$$

- SL(2,R) Zamolodchikov relation looks simpler.
- Is there any relation? One observes that

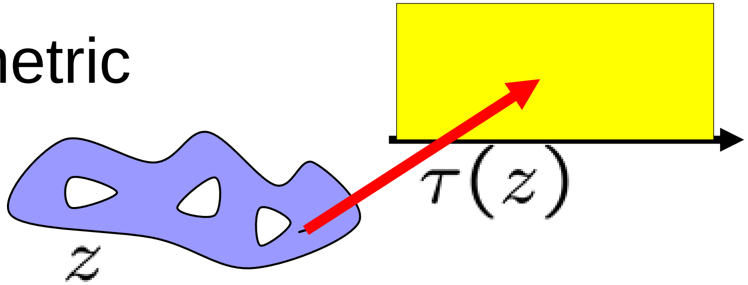
$$\varphi(z|x) \equiv -2\ln(\frac{\pi}{2}\Phi_2) \quad \partial_x \partial_{\bar{x}} \varphi(z|x) = -2e^{\varphi(z|x)}$$

- Appearance of Liouville equation of motion!

USO (Uniformizing Schwarzian Operator)

- Consider Riemann surface with metric

$$e^{\varphi} dz d\bar{z} = \frac{|\tau'|^2}{(\mathrm{Im}\tau)^2} dz d\bar{z}$$



- Liouville theorem: τ is inverse of uniformization map.
- Uniformizing Schwarzian Operator is defined as

$$D_m = \mathcal{S}_{\tau}^{(m)} = \tau'^{(m-1)/2} \underbrace{\partial_z \tau'^{-1} \ldots \partial_z \tau'^{-1} \partial_z}_{m\, derivatives} \tau'^{(m-1)/2} ,$$

- Theorem: USO is invariant under SL(2,C) transform of τ

$$\tau \to \frac{A\tau + B}{C\tau + D} , \qquad \tau' \to (C\tau + D)^{-2} \tau'$$

- Metric is only invariant under SL(2,R).

Crucial observation

- ∃ SL(2,C) such that $\tau' \to e^{\varphi}$
- We choose

$$C = \frac{1}{2i\bar{\tau}'^{1/2}} \quad D = \frac{-\bar{\tau}}{2i\bar{\tau}'^{1/2}}$$

- This SL(2,C) depends on $\bar{z}$, but USO is invariant.

$$S_{\tau}^{(m)} = e^{\frac{m-1}{2}\varphi}\partial_z e^{-\varphi} \cdots e^{-\varphi}\partial_z e^{\frac{m-1}{2}\varphi}$$

- Theorem: $D_m = S_{\partial_{\bar{z}}\varphi}^{(m)} = S_{\tau}^{(m)}$

Zamolodchikov relation from USO

- Derivation of Zamolodchikov relation is much simpler by using USO.

$$\bar{S}_\tau^{(m)} S_\tau^{(m)}[\varphi e^{-\frac{1-m}{2}\varphi}] = 2(-1)^{m+1} 4^{-m} m!(m-1)! e^{\frac{m+1}{2}\varphi}$$

- To derive this, it is important to realize

$$S_\tau^{(m)} = \left(\frac{\partial z}{\partial \tau}\right)^{-\frac{m+1}{2}} \partial_\tau^m \left(\frac{\partial z}{\partial \tau}\right)^{\frac{1-m}{2}}$$

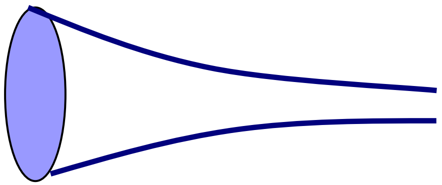
$$\varphi e^{\frac{1-m}{2}\varphi} = -\left(\ln\left|\frac{\partial z}{\partial \tau}\right|^2 + 2\ln y\right) y^{m-1} \left|\frac{\partial z}{\partial \tau}\right|^{m-1}, \quad y = \mathrm{Im}\tau$$

- The result is the same as Zamolodchikov with $\mu^c = \frac{1}{2}$

Hidden Liouville equation in SL(2,R)

- Consider SL(2,R) WZNW model

$$ds^2 = d\phi^2 + e^{2\phi} d\gamma d\bar{\gamma}$$



- Degenerate operator

$$\Phi_m(z|x) = \frac{m}{\pi}\left(\frac{\pi}{2}\Phi_2\right)^{m-1}$$

$$\Phi_2(z|x) = \frac{2}{\pi}(|x-\gamma(z)|^2 e^{\phi(z)} + e^{-\phi(z)})$$

- Hidden Liouville equation

$$\varphi(z|x) \equiv -2\ln(\frac{\pi}{2}\Phi_2) \qquad \partial_x \partial_{\bar{x}} \varphi(z|x) = -2e^{\varphi(z|x)}$$

- Actually x is uniformizing (trivializing) coordinate

$$T(x) \equiv 0$$

- USO is just a partial derivative! Origin of simplicity.

$$D_m = S^{(m)} = \partial_x^m$$

An application to AdS3/CFT2

- Hidden Liouville equation in SL(2,R) yields a set of Ward-Takahashi identity for boundary CFT.

$$\partial_x \partial_{\bar{x}} \varphi(z|x) = -2e^{\varphi(z|x)}$$

$$\partial_x^m \partial_{\bar{x}}^m \tilde{\Phi}_m = -m!(m-1)!\Phi_{-m}$$

- AdS/CFT correspondence

$$\left\langle \prod_i \Phi_{j_i}(x_i, \bar{x}_i) \right\rangle_{BCFT} = \left\langle \prod_i \int d^2 z_i \Phi_{j_i}(z_i, \bar{z}_i | x_i, \bar{x}_i) \right\rangle_{ws}$$

- Infinitely many Zamolodchikov equation on RHS gives nontrivial constraint on the CFT correlator on LHS.

- Possible complete solvability of AdS3/CFT2 model?

Summary 1

- Zamolodchikov relation for Liouville theory is related to USO.
- USO becomes simple in the trivializing coordinate (uniformizing coordinate).
- Zamolodchikov relation for SL(2,R) WZNW model is realized in such a coordinate.
- Possible application to AdS/CFT.

Stoyanovsky-Ribault-Teschner map and string scattering amplitudes

A. V. Stoyanovsky: math-ph/0012013 (withdrawn)

S. Ribault, J. Teschner: hep-th/0502048

Stoyanovsky-Ribault-Teschner map

- N-pt function in SL(2,R) ~ 2N-2 pt function in Liouville

$$\langle \Phi_{j_1,m_1,\bar{m}_1} \cdots \Phi_{j_N,m_N,\bar{m}_N} \rangle^{SL(2,R)}$$

$$\sim \langle \prod_{t=1}^{2N-2} V_{\alpha_t}(z_t) \rangle^{Liouville}$$

- Application to string theory on AdS_3, 2D BH, tachyon condensation.
- Many non-perturbative effects in string theory will be understood from Liouville theory through SRT map.

The Formula (SRT map)

$$\langle \prod_{i=1}^{N} \Phi_{j_i,m_i,\bar{m}_i}(z_i) \rangle^H = \prod_{i=1}^{N} N^{j_i}_{m_i,\bar{m}_i} \prod_{r=N+1}^{2N-2} \int d^2 z_r F_k(z) \times$$

$$\times \langle \prod_{t=1}^{N} V_{\alpha_t}(z_t) \prod_{r=N+1}^{2N-2} V_{\alpha_{1,2}}(z_r) \rangle^L$$

With

$$F_k(z) \sim \mu^{-1+b^{-1}\sum \alpha - b^{-1}} \prod (z_r - z_l)^{m_r+\bar{m}_r+k/2} \ldots$$

Leg factor:

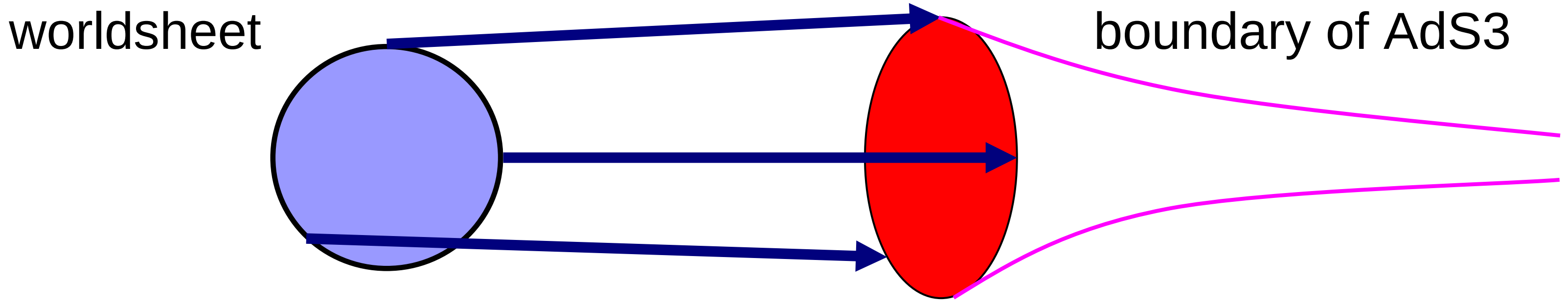
$$N^j_{m,\bar{m}} = -\frac{\Gamma(-j+m)}{\Gamma(1+j-\bar{m})}$$

Parameter map:

$$\alpha_r = b j_r + b + b^{-1}/2; \quad b^{-2} = k - 2$$

Instanton in AdS3

- Instanton contribution is encoded as bulk poles (Liouville part).



Existence of holomorphic map $\gamma(z) = z^{\omega}$

Divergence at $k + N - 3 + \sum_{i=1}^{N} j_i = 0$

Liouville theory correlation has bulk poles at

$$\sum_{i=1}^{N} \alpha_i + n_+ b + n_- b^{-1} = Q$$

Under SRT map, they totally agree (w=1 → n- = 1)

LSZ in LST

- Gauge/Gravity correspondence: String ON-shell = Gauge OFF-shell
- Parameter tune of String → Gauge theory ON-shell!

- Green function should have LSZ poles (Aharony et al hep-th/0404016).

$$\langle O_1(p_1)\dots O_n(p_n)\rangle \sim \prod_i \frac{1}{p_i^2+M_i^2}\langle 0|O_1^{(norm)}(p_1)\dots O_n^{(norm)}(p_n)|0\rangle$$

- Each vertex should contain such poles. (p ~ j, M ~ m)

- SRT map:

$$N^j_{m,\bar{m}} = -\frac{\Gamma(-j+m)}{\Gamma(1+j-\bar{m})}$$

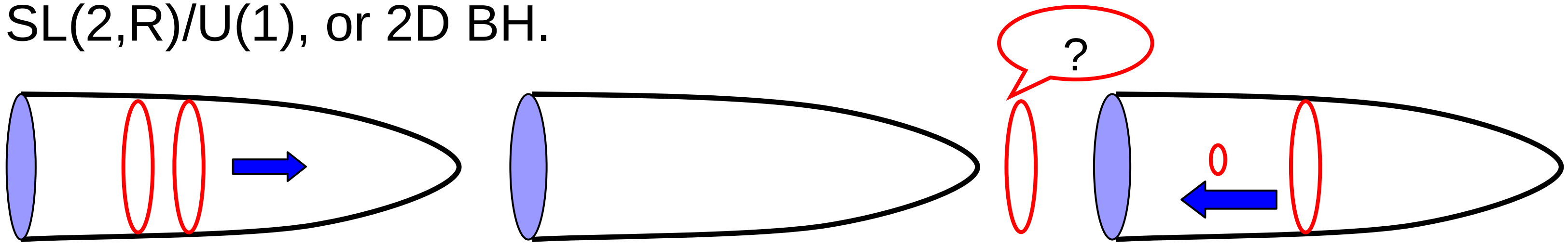
- Expected poles at

$$j = M-1, M-2, \dots > -\frac{1}{2}, M = \min\{|m|, |\bar{m}|\}$$

- RST map can be seen as LSZ reduction.

Winding violating correlator

- FZZ (unpublished) computed winding violating correlator in SL(2,R)/U(1), or 2D BH.



- Following FZZ, we introduce conjugate representation of identity operator (spectral flow) $\Phi_{-k/2,-k/2,-k/2}$
- Under SRT map, it becomes just 1.

$$\langle \Phi_{j_1,m_1,\bar{m}_1} \Phi_{j_2,m_2,\bar{m}_2} \Phi_{j_3,m_3,\bar{m}_3} \rangle^{winding} \sim$$

$$\sim \langle \Phi_{j_1,m_1,\bar{m}_1} \Phi_{-k/2,-k/2,-k/2} \Phi_{j_2,m_2,\bar{m}_2} \Phi_{j_3,m_3,\bar{m}_3} \rangle^{WZNW}$$

$$\sim \prod_{i=1}^{3} N^{j_i}_{m_i,\bar{m}_i} \int d^2 z_4 d^2 z_5 F(z) \langle \prod_{i=1}^{3} V_{\alpha_i} \prod_{l=4}^{5} V_{-1/2b}(z_l) \rangle^{Liouville}$$

- In general, M violating N-pt amplitudes become 2N+M-2 pt function.

Explicit winding violating amplitudes (3pt)

- FZZ gives the explicit amplitudes in terms of elementary functions.

$$\sim \frac{\Gamma(1+j_1+j_2+j_3+k/2)}{\Gamma(-j_1-j_2-j_3-k/2)} \prod_{i=1}^{3} \frac{\Gamma(-j_i+m_i)}{\Gamma(1+j_i-\bar{m}_i)} \times B(j_1) C^H(-k/2-j_1, j_2, j_3)$$

- Our formula captures several features without calculation.
- LSZ pole explains group factor: $\prod_i N^{j_i}_{m_i,\bar{m}_i} = \prod_i \frac{\Gamma(-j_i+m_i)}{\Gamma(1+j_i-\bar{m})}$
- Liouville 5pt function (in the leading order singularity) explains structure const:

$$\langle \prod_{i=1}^{3} V_{\alpha_i} V_{-1/2b} V_{-1/2b} \rangle^{Liouville} \sim C^H(-k/2-j_1, j_2, j_3)$$

Sine Liouville becomes Liouville at k = 0 (c=-2).

- FZZ dualtiy → Sine Liouville = SL(2,R)/U(1) WZNW.

$$S = \frac{1}{2\pi}\int d^2z(\partial\varphi\bar{\partial}\varphi + \partial X\bar{\partial}X - \frac{1}{2\sqrt{k-2}}R\varphi + e^{-\sqrt{\frac{k-2}{2}}\varphi}\cos\sqrt{k}X$$

- If we set k=0, is it Liouville (at $b^2 = -1/2$) with free boson X?
- This system can be seen as the toy model of closed string tachyon condensation (Hikida-Takayanagi).
- SRT map answers this question. YES!

$$\langle\prod_{i=1}^{N}\Phi_{j_i,0,0}(z_i)\rangle^{SL} = \prod_{i=1}^{N}R_0(j_i)\prod_{t=1}^{N-2}\int d^2v_t$$

$$\langle\prod_{i=1}^{N}V_{bj_i}(z_i)\prod_{t=1}^{N-2}V_b(v_t)\rangle^{Liouville}$$

- Reflection amplitudes come from our convention.

$$R_0(j) = 2^{2j+2}\Gamma(-j)/\Gamma(1+j)$$

Summary 2

- SRT map enabled us to understand important features of string scattering amplitudes on curved background.

- World sheet instanton effects in AdS_3.
  - Bulk poles from dual Liouville screening
- LSZ reduction mechanism in LST.
  - Complete separation of LSZ poles from bulk poles
- Winding violating amplitudes in 2D BH.
  - M violating N pt function ~ 2N+M-2 pt function in Liouville
- $k \to 0$ limit of FZZ conjecture and tachyon condensation.
  - Agreeing with the naïve action level arguments.

Open question and loose end

- Can we understand Zamolodchikov's higher equation of motions from SRT map?

- SRT map for $\tilde{j}^{-}_{m,n} = \frac{m-1}{2} - \frac{n}{2}(k-2)$ we have

$$\tilde{\alpha}_{m,n} = \frac{m+1}{2} b - \frac{n-1}{2b} = b\tilde{j}^{-}_{m,n} + b + \frac{b^{-1}}{2}$$

- Zamolodchikov's coefficient seems to agree.

- However, the operation on WZNW looks nontrivial on Liouville side.