Four-Point Function in Minimal Liouville Gravity

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1 Preliminaries

- **Liouville gravity** (LG) – 2D quantum gravity with effective action induced by a "critical" matter, i.e., CFT $\mathcal{M}_c$ with central charge c. Induced action – *universal Liouville action* (A.Polyakov, 1981).

  $\{\Phi_i, \Delta_i\}$ – set of primaries and their dimensions in $\mathcal{M}_c$.

- **Liouville field theory** (LFT) – QFT based on the *Liouville action.* LFT –> non-rational CFT with $c_{\mathrm{L}} = 1 + 6Q^2$; $Q = b^{-1} + b$; b – parameter

$$\mathcal{L}_{\mathrm{L}} = \frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi}$$

  μ – cosmological constant

  quantized metric — $ds^2 = e^{2b\phi} \hat{g}_{ab} dx^a dx^b$ ($\hat{g}_{ab}$ – "background metric")

  – Primary fields – continuous family of "exponentials" $V_a = \exp(2a\phi)$, a – (complex) parameter

$$\Delta_a^{(\mathrm{L})} = a(Q-a)$$

— Exactly solvable (H.Dorn & H.Otto, 1992): explicit OPE structure contants (3-p function)

$$C^{(\mathrm{L})}_{a_1,a_2,a_3} = \langle V_{a_1}(x_1)V_{a_2}(x_2)V_{a_3}(x_3)\rangle_{\mathrm{L}}$$

— In LG b is tuned as: $c + c_{\mathrm{L}} = 26$

- **Reparametrization ghost** field theory: BC system $(2,-1)$

$$A_{\mathrm{gh}} = \frac{1}{\pi}\int (C\bar{\partial}B + \bar{C}\partial\bar{B})d^2x$$

with $c_{\mathrm{gh}} = -26$.

– matter + Liouville T central charge 26; form BRST complex w.r.t. nilpotent

$$\mathcal{Q} = \oint (CT + C\partial CB)\,\frac{dz}{2\pi i}$$

- **Correlation functions**: matter operators Φ_i are "dressed" by appropriate V_{a_i} to form

– either $(1,1)$ form $U_i = \Phi_i V_{a_i}$ (ghost # 0)

– or $(0,0)$ scalar $W_i = C\bar{C}U_i$ (ghost $\# = 1$)

$$\Delta_i + a_i(Q - a_i) = 1$$

– Gauge inv. corr. functions –> correlation numbers (CN). Genus 0 n-point CN

$$\langle U_1 \dots U_n \rangle_{\mathrm{G}} = \int_{M_n} \langle W_1(x_1) \dots W(x_n) \rangle =$$

$$\int \left\langle W_1 W_2 W_3 U_4(x_4) d^2 x_4 \dots U(x_n) d^2 x_n \right\rangle$$

M_n – $n-3$ dim. moduli space of sphere with n punctures.

– 3p : no moduli integration. Factorized

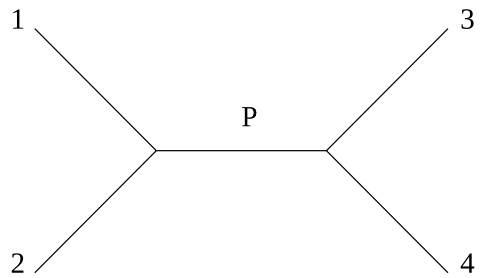
$$\langle U_1 U_2 U_3 \rangle_{\mathrm{G}} = |x_{12} x_{23} x_{31}|^2 \times$$
$$\langle \Phi_1(x_1) \Phi_2(x_2) \Phi_3(x_3) \rangle \langle V_1(x_1) V_2(x_2) V_3(x_3) \rangle$$

- **Four point CN** : one moduli integration

$$\langle U_1 U_2 U_3 U_4 \rangle_{\mathrm{G}} = |x_{12} x_{23} x_{31}|^2 \times$$
$$\int \langle \Phi_1(x_1) \dots \Phi_4(x_4) \rangle \langle V_1(x_1) \dots V_4(x_4) \rangle \, d^2 x_4$$

– general Liouville 4-p function

$$\langle V_1(x_1) \dots V_4(x_4) \rangle =$$
$$\int \frac{dP}{4\pi} C^{(\mathrm{L})}_{a_1, a_2, Q/2+iP} C^{(\mathrm{L})Q/2+iP}_{a_3, a_4} \; \mathcal{F}_P(x) \mathcal{F}_P(\bar{x})$$

$\mathcal{F}_P(x) =$ 

– general 4p conformal block (BPZ 1984),

– less transparent then the 3p CN

- **(Generalized) minimal gravity** (GMG): matter CFT = (generalized) minimal model (GMM) $\mathcal{M}_{b^2}$. If one $\Phi_i = \Phi_{m,n}$ – degenerate matter field – 4p modular integral can be evaluated due to higer equations of motion (HEM) in Liouville.

- **4p CN in GMG with degenerate field**

$$G_{m,n}(\alpha_1, \alpha_2, \alpha_3) = \langle U_{\alpha_1} U_{\alpha_2} U_{\alpha_3} U_{m,n} \rangle_{\text{GMG}}$$

$U_{m,n} = \Phi_{m,n}\tilde{V}_{m,n}$, $\tilde{V}_{m,n}$ – appropriate LFT dressing, U_{α_i} – *GENERIC* non-degenerate. HEM: integrand —> derivative, integral –> boundary terms + curvature term.

Curvature term –> $\langle O_{m,n} W_1 W_2 W_3 \rangle$ of

- **Discrete state** $O_{m,n}$ (I.Klebanov and A.Polyakov, 1991; E.Witten, 1992) related to $\Phi_{m,n}$ and the corresponding degenerate LFT $V_{m,n}$.

2 Generalized minimal models

- **"Canonic"** $\mathcal{M}_{p/p'}$ – rational CFT with $(p-1)(p'-1)/2$ (degenerate) primary fields $\Phi_{m,n}$; $(m,n) = (1 : p-1, 1 : p'-1)$ and $\Phi_{p-m,p'-n} = \Phi_{m,n}$.

- **GMM** $\mathcal{M}_{b^2}$ – formal CFT

  – continuous $b^2 = p/p'$

$$c_{\mathrm{M}} = 1 - 6(b^{-1} - b)^2$$

  – continuous spectrum of primaries Φ_α with $(q = b^{-1} - b)$

$$\Delta_\alpha^{(\mathrm{M})} = \alpha(\alpha - q)$$

  – Normalization $\langle \Phi_\alpha \Phi_\alpha \rangle_{\mathrm{GMM}} = (x\bar{x})^{-2\Delta_\alpha}$

  – Degenerate fields $\Phi_{m,n}$ $(\alpha = q/2 \pm \lambda_{m,-n})$

$$\Delta_{m,n}^{(\mathrm{M})} = -q^2/4 + \lambda_{m,-n}^2$$

  – notation

$$\lambda_{m,n} = mb^{-1}/2 + nb/2$$

- **Singular vectors** in $\Phi_{m,n}$ rep. vanish

$$D_{m,n}^{(\mathrm{M})} \Phi_{m,n} = \bar{D}_{m,n}^{(\mathrm{M})} \Phi_{m,n} = 0$$

$D^{(\mathrm{M})}_{m,n}$ – “singular vector creating” operator made of Virasoro generators M_n

$$D^{(\mathrm{M})}_{1,2} = M^2_{-1} - b^2 M_{-2}$$
$$D^{(\mathrm{M})}_{1,3} = M^3_{-1} - 2b^2\{M_{-2}M_{-1}\} + 4b^4 M_{-3}$$
$$\ldots$$

====>

- **Degenerate OPE** – finite # of terms, e.g.

$$\Phi_{1,2}(x)\Phi_\alpha(0) = C^{(\mathrm{M})}_+(\alpha)(x\bar{x})^{\alpha b}\left[\Phi_{\alpha+b/2}\right] + C^{(\mathrm{M})}_-(\alpha)(x\bar{x})^{1-\alpha b-b^2}\left[\Phi_{\alpha-b/2}\right]$$

in CFT normalization

$$C^{(\mathrm{M})}_+(\alpha) = \left[\frac{\gamma(b^2)\gamma(2\alpha b+2b^2-1)}{\gamma(2b^2-1)\gamma(b^2+2\alpha b)}\right]^{1/2}$$
$$C^{(\mathrm{M})}_-(\alpha) = \left[\frac{\gamma(b^2)\gamma(2\alpha b+b^2-1)}{\gamma(2b^2-1)\gamma(2\alpha b)}\right]^{1/2}$$

- **Degenerate** $\Phi_{m,n}$ 4-p correlation function

$$G^{(\mathrm{GMM})}_{(m,n),\alpha_1,\alpha_2,\alpha_3}(x) = \sum_{r,s}^{(m,n)} C^{(\mathrm{M})}_{r,s}(\alpha_1)\times C_{\mathrm{M}}(\alpha_1+\lambda_{r,-s},\alpha_2,\alpha_3)\mathcal{F}_{r,s}(x)\mathcal{F}_{r,s}(\bar{x})$$

– finite number mn of bloks $\mathcal{F}_{r,s}(x)$ with $(r,s) = (-m+1:2:m-1, -n+1:2:n-1)$

– IMPORTANT: presently all remaining 3 α_i are *generic non-degenerate.*

– ATTENTION: Naive limit $\alpha_i \to \alpha_{m,n}$ degenerate isn't always correct

3 Higher equations of motion

- **Degenerate Liouville** exponentials $V_{m,n} = V_{a_{m,n}}$ with $a_{m,n} = Q/2 - \lambda_{m,n}$. Kac dimensions for $c_{\rm L}$

$$\Delta^{(\rm L)}_{m,n} = Q^2/4 + \lambda^2_{m,n} = 1 - \Delta^{(\rm M)}_{m,n} - mn$$

- **Singular vectors vanish** in LFT

$$D^{(\rm L)}_{m,n} V_{m,n} = \bar{D}^{(\rm L)}_{m,n} V_{m,n} = 0$$

$D^{(\rm L)}_{m,n}$ – "singular vector creating" operators in Liouville made of L_n – Liouville Virasoro generators:

$$D^{(\rm L)}_{m,n} = D^{(\rm M)}_{m,n} \;/.\; \left\{ b^2 \to -b^2,\ M_n \to L_n \right\}$$

- **"Logarithmic degenerate"** fields

$$V'_{m,n} = \frac{1}{2}\frac{\partial}{\partial a} V_a|_{a=a_{m,n}} = \phi \exp(2a_{m,n}\phi)$$

inhomogeneous transformation

$$|y_x|^{2\Delta_{m,n}}\, V'_{m,n}(y) =$$
$$V'_{m,n}(x) - \Delta'_{m,n} V_{m,n}(x) \log|y_x|$$

with

$$\Delta'_{m,n} = \frac{d}{da}\Delta_a^{(\mathrm{L})}|_{a=a_{m,n}} = mb^{-1} + nb$$

- **HEM**

$$D^{(\mathrm{L})}_{m,n} \bar{D}^{(\mathrm{L})}_{m,n} V'_{m,n} = B_{m,n} \tilde{V}_{m,n}$$

$\tilde{V}_{m,n} = V_a|_{a=a_{m,-n}}$ – LFT exponential

$$\tilde{\Delta}^{(\mathrm{L})}_{m,n} = \Delta^{(\mathrm{L})}_{m,n} + mn = 1 - \Delta^{(\mathrm{M})}_{m,n}$$

the "dressing" Liouville field for matter degenerate $\Phi_{m,n}$

$$U_{m,n} = \Phi_{m,n} \tilde{V}_{m,n}\ ;\ \dim = (1,1)$$

Coefficient

$$B_{m,n} = \frac{\left(\pi\mu\gamma(b^2)\right)^n b^{1+2n-2m}}{\gamma(1-m+nb^2)} \prod_{k,l}^{\{m,n\}} 2\lambda_{k,l}$$

$$(k,l) = (-m+1 : m-1, -n+1 : n-1) \backslash (0,0)$$

4 Generalized minimal gravity

- **Dressed fields**

$$U_a = \Phi_{a-b} V_a$$

– second solution $U_{Q-a} = \Phi_{a-b} V_{Q-a}$.

- **Three-point CN**

$$\langle W_{a_1} W_{a_2} W_{a_2} \rangle_{\mathrm{GMG}} = \Omega \prod_{i=1}^{3} N(a_i)$$

with

$$\Omega = \left[\pi\mu\gamma(b^2)\right]^{Q/b} \left[\gamma(b^2)\gamma(b^{-2}-1)b^{-2}\right]^{1/2}$$

– "leg-factors"

$$N(a) = \left[\frac{\gamma(2ab-b^2)\gamma(2ab^{-1}-b^{-2})}{(\pi\mu\gamma(b^2))^{2a/b}}\right]^{1/2}$$

– Two point CN

$$\langle U_a U_a \rangle_{\mathrm{GMG}} = \left[\pi\mu\gamma(b^2)\right]^{Q/b} \frac{N^2(a)}{\pi(2a-Q)}$$

– partition function

$$Z_{\mathrm{L}} = \left[\pi\mu\gamma(b^2)\right]^{Q/b} \frac{1-b^2}{\pi^3 Q \gamma(b^2)\gamma(b^{-2})}$$

- **Normalized CN's**

$$\langle\langle U_1 \dots U_n \rangle\rangle = Z_{\rm L}^{-1} \, \langle U_1 \dots U_n \rangle_{\rm GMG}$$

– "normalized leg-factors"

$$\mathcal{N}(a) = \left[\frac{\gamma(2ab - b^2)\gamma(2ab^{-1} - b^{-2})}{(\pi\mu)^{2a/b}\gamma^{2a/b-1}(b^2)\gamma(2 - b^{-2})} \right]^{1/2}$$

– *"gravitational normalization"* of dressed fields

$$\mathcal{U}(a) = \mathcal{N}^{-1}(a) U_a \; ; \quad \mathcal{W}(a) = \mathcal{N}^{-1}(a) W_a$$

2-p and 3-p normalized CN's

$$\langle\langle \mathcal{U}(a)\mathcal{U}(a) \rangle\rangle = \frac{\pi^2 (g+1) g (g-1)}{g + 1 - 2s}$$

$$\langle\langle \mathcal{U}_{a_1}\mathcal{U}_{a_2}\mathcal{U}_{a_3} \rangle\rangle = -\pi^3 (g+1) g (g-1)$$

Moreover

$$\mathcal{U}(a) = \mathcal{U}(Q - a)$$

i.e., independent on the choice of Liouville "dressing". Price = singularities in the leg-factors + dependence on μ.

5 4p integral and discrete states

Let

$$\mathcal{D}_{m,n} = D^{(\rm M)}_{m,n} + (-)^{mn} D^{(\rm L)}_{m,n}$$

and

$$\Theta_{m,n} = \Phi_{m,n} V_{m,n}$$

Statement 1: *For every pair* (m,n) of *positive integers there is a graded polynomial* $H_{m,n}$ *in generators* M_n, L_n *and ghosts* B, C *of order* $mn-1$ *and ghost number* 0 *(unique* mod *exact terms) such that* $H_{m,n}\Theta_{m,n}$ *is closed and nontrivial.*

One finds explicitly

$$\begin{aligned} H_{1,2} &= M_{-1} - L_{-1} + b^2 CB \\ H_{1,3} &= M_{-1}^2 - M_{-1}L_{-1} + L_{-1}^2 - 2b^2 \left(M_{-2} + L_{-2}\right) + \\ &\quad + 2b^2 \left(M_{-1} - L_{-1}\right) CB - 4b^4 C\partial B \end{aligned}$$

$(m,n) = (1,n)$ proof: C.Imbimbo, S.Mahapatra and S.Mukhi. Nucl.Phys., B **375** (1992) 399. General statement is most likely also true (B.Feigin, private communication).

– Normalization of $H_{m,n}$

$$H_{m,n} = \sum_{k=0}^{mn-1} \left(M_{-1}\right)^{mn-1-k} \left(-L_{-1}\right)^k + \ldots.$$

– Apparently

$$\left(\partial H_{m,n} - \mathcal{Q} R_{m,n}\right) \Theta_{m,n} = 0$$

$R_{m,n}$ graded polynomial of order mn, ghost # -1.

– Let

$$\Theta'_{m,n} = \Phi_{m,n} V'_{m,n}$$

Statement 2:

$$\mathcal{D}_{m,n}\bar{\mathcal{D}}_{m,n}\Theta'_{m,n} =$$
$$\left(\partial H_{m,n} - \mathcal{Q}R_{m,n}\right)\left(\bar{\partial}\bar{H}_{m,n} - \bar{\mathcal{Q}}\bar{R}_{m,n}\right)\Theta'_{m,n}$$

Verified directly for $(m,n) = (1,2)$ and $(m,n) = (1,3)$. General $(m,n) = ?$

- **4-p CN** $\langle\langle U_{m,n}U_{\alpha_1}U_{\alpha_2}U_{\alpha_3}\rangle\rangle$ of $U_{m,n}$

$$Z_{\mathrm{L}}^{-1}\int \langle U_{m,n}(x)W_{a_1}W_{a_2}W_{a_3}\rangle_{\mathrm{GMG}}\, d^2x$$
$$= B_{m,n}^{-1}\int \partial\bar{\partial}\left\langle\left\langle O'_{m,n}(x)W_{a_1}W_{a_2}W_{a_3}\right\rangle\right\rangle d^2x$$

– here

$$O'_{m,n} = H_{m,n}\bar{H}_{m,n}\Theta'_{m,n}$$

Integral —> Boundary terms near $W_{a_i}(x_i)$ insertions + "curvature term". In "grav. normalization"

$$\langle\langle \mathcal{U}_{m,n}\mathcal{U}_{\alpha_1}\mathcal{U}_{\alpha_2}\mathcal{U}_{\alpha_3}\rangle\rangle = \mathcal{N}(a_{m,-n})B_{m,n}^{-1}\times$$
$$\int \left\langle\left\langle \partial\bar{\partial}O'_{m,n}(x)\mathcal{W}_{a_1}\mathcal{W}_{a_2}\mathcal{W}_{a_3}\right\rangle\right\rangle d^2x$$

6 Curvature term

Field $O'_{m,n}(x)$ is not exactly a $(0,0)$ form but

$$O'_{m,n}(y) = O'_{m,n}(x) - 2\lambda_{m,n} O_{m,n}(x) \log |y_x|$$

where

$$O_{m,n} = H_{m,n} \bar{H}_{m,n} \Theta_{m,n}$$

a *discrete state* (I.Klebanov and A.Polyakov, 1991), i.e., a physical state of ghost # 0. Discrete states form algebra called the *ground ring* (E.Witten, 1992).

Let $\widehat{g}_{ab} = e^{\sigma} \delta_{ab}$ – a background metric. Then

$$\sigma(y) = \sigma(x) - 2 \log |y_x|$$

hence

$$\tilde{O}'_{m,n}(x) = O'_{m,n}(x) - \lambda_{m,n} \sigma(x) O_{m,n}(x)$$

is a scalar. Covariant HEM reads

$$B_{m,n} U_{m,n} = \frac{1}{4} \sqrt{\widehat{g}} \left(\widehat{\triangle} \tilde{O}'_{m,n} - \lambda_{m,n} \widehat{R} O_{m,n} \right) + \text{exact}$$

$\widehat{\triangle}$ – covariant Laplace operator in $\widehat{g}_{ab}$, $\widehat{R}$ – scalar curvature. Second term gives

$$\int \langle\langle \partial\bar{\partial} O'_{m,n} U_{\alpha_1} U_{\alpha_2} U_{\alpha_3} \rangle\rangle \, d^2x =$$

$$- 2\pi \lambda_{m,n} \langle\langle O_{m,n} U_{\alpha_1} U_{\alpha_2} U_{\alpha_3} \rangle\rangle + \text{b.t.}$$

7 Ground ring in GMG

Discrete states $O_{m,n}$ act in the space of classes W_a. From fusion rules for $\Phi_{m,n}$ and $V_{m,n}$

$$O_{m,n}W(a) = \sum_{r,s}^{(m,n)} A_{r,s}^{(m,n)} W(a+\lambda_{r,s}) + \text{exact}$$

Evaluate the coefficients $A_{r,s}^{(m,n)}$. For $(m,n) = (1,2)$

$$V_{1,2}(y)V_a(0) = C_+^{(\mathrm{L})}(a)(y\bar{y})^{ab}\left[V_{a-b/2}\right] + \\ + C_-^{(\mathrm{L})}(a)(y\bar{y})^{1-ab+b^2}\left[V_{a+b/2}\right]$$

Combine with

$$\Phi_{1,2}(x)\Phi_\alpha(0) = C_+^{(\mathrm{M})}(\alpha)(x\bar{x})^{\alpha b}\left[\Phi_{\alpha+b/2}\right] \\ + C_-^{(\mathrm{M})}(\alpha)(x\bar{x})^{1-\alpha b-b^2}\left[\Phi_{\alpha-b/2}\right]$$

Acting by $H_{12} = \partial_x - \partial_y + b^2 CB$ deletes "wrong terms" while decorates "good" ones with multipliers $\left(1-2ab+b^2\right)$

$$A_{0,-1}^{(1,2)} = \left(1-2ab+b^2\right)^2 C_-^{(\mathrm{M})}(a-b)C_+^{(\mathrm{L})}(a) \\ A_{0,1}^{(1,2)} = \left(1-2ab+b^2\right)^2 C_+^{(\mathrm{M})}(a-b)C_-^{(\mathrm{L})}(a)$$

Result reads in general

$$A_{r,s}^{(m,n)} = \frac{\Lambda_{m,n}\mathcal{N}(a)}{\mathcal{N}(a+\lambda_{r,s})}$$

with

$$\Lambda_{m,n} = b^{-1} B_{m,n} \mathcal{N}(a_{m,-n})$$

and $B_{m,n}$ are those entering HEM. Let $\mathcal{O}_{m,n} = \Lambda^{-1}_{m,n} O_{m,n}$

$$\mathcal{O}_{m,n} \mathcal{W}(a) = \sum_{r.s}^{(m,n)} \mathcal{W}(a + \lambda_{r,s})$$

- **The ring structure** follows

$$\mathcal{O}_{m,n} \mathcal{O}_{m',n'} = \sum_{l}^{[m,m']} \sum_{k}^{[n,n']} \mathcal{O}_{l,k}$$

where $\sum_k^{[n,n']}$ implies sum over $k = \min(|n - n'|, 0) : 2 : n + n'$ (Cp. N.Seiberg & D.Shih, 2003)

8 Boundary terms

- **Controlled by OPE** of the logarithmic field $O'_{m,n}(x)$ with a field W_a. Study OPE $\Theta'_{m,n} = \Phi_{m,n} V'_{m,n}$ with $C\bar{C}\Phi_{a-b} V_a$. Basically, relevant terms in

$$V'_{m,n}(y) V_a(0) = \ldots$$

- **Discrete degenerate OPE**

  $$V_{m,n}(y)V_a(0) = \sum_{r,s}^{(m,n)} C^{(m,n)}_{r,s}(a)(y\bar{y})^{\Delta^{(\mathrm{L})}_{a+\lambda_{r,s}}-\Delta^{(\mathrm{L})}_{a}-\Delta^{(\mathrm{L})}_{m,n}} \left[V_{a+\lambda_{r,s}}\right]$$

  from the singular vector decoupling.

- **General Liouville OPE**

  $$V_g(x)V_a(0) = \int_{\uparrow} \frac{dp}{4\pi i} C^{(\mathrm{L})p}_{g,a}(x\bar{x})^{\Delta^{(\mathrm{L})}_{p}-\Delta^{(\mathrm{L})}_{g}-\Delta^{(\mathrm{L})}_{a}} [V_p(0)]$$

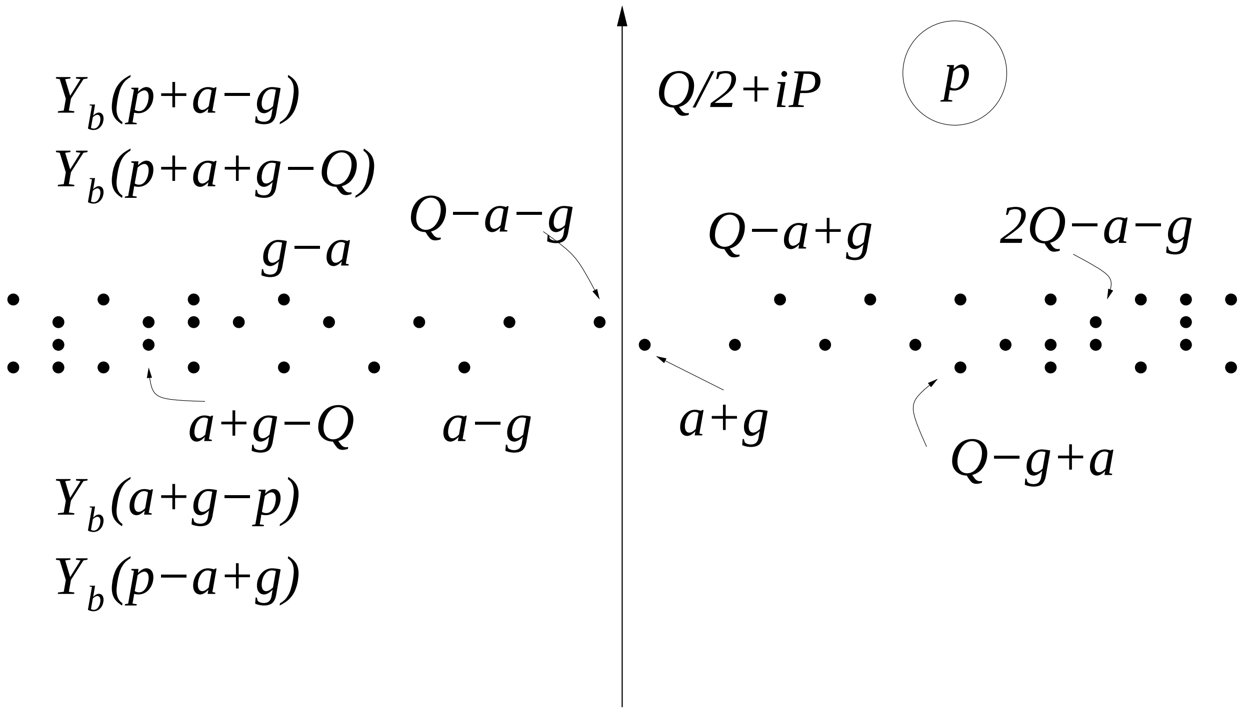
  with

  $$C^{(\mathrm{L})p}_{g,a} = \frac{\Upsilon_b(2g)(\pi\mu\gamma(b^2)b^{2-2b^2})^{(p-a-g)}}{\Upsilon_b(p+g-a)\Upsilon_b(a+g-p)} \times \frac{\Upsilon_b(b)\Upsilon_b(2a)\Upsilon_b(2Q-2p)}{\Upsilon_b(p+a-g)\Upsilon_b(a+g+p-Q)}$$

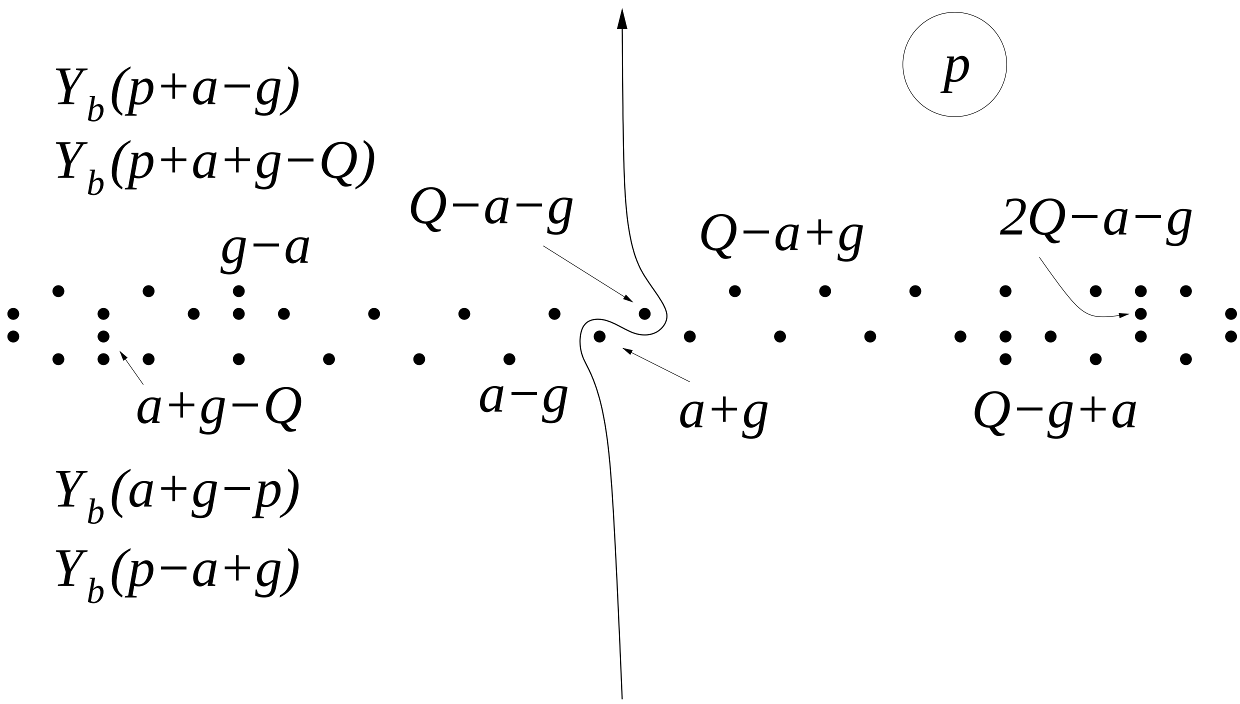
  (see picture)

- **Discrete terms.** Let $g \to 0$. Poles at $p = a+g$ and $p = Q-a-g$ give

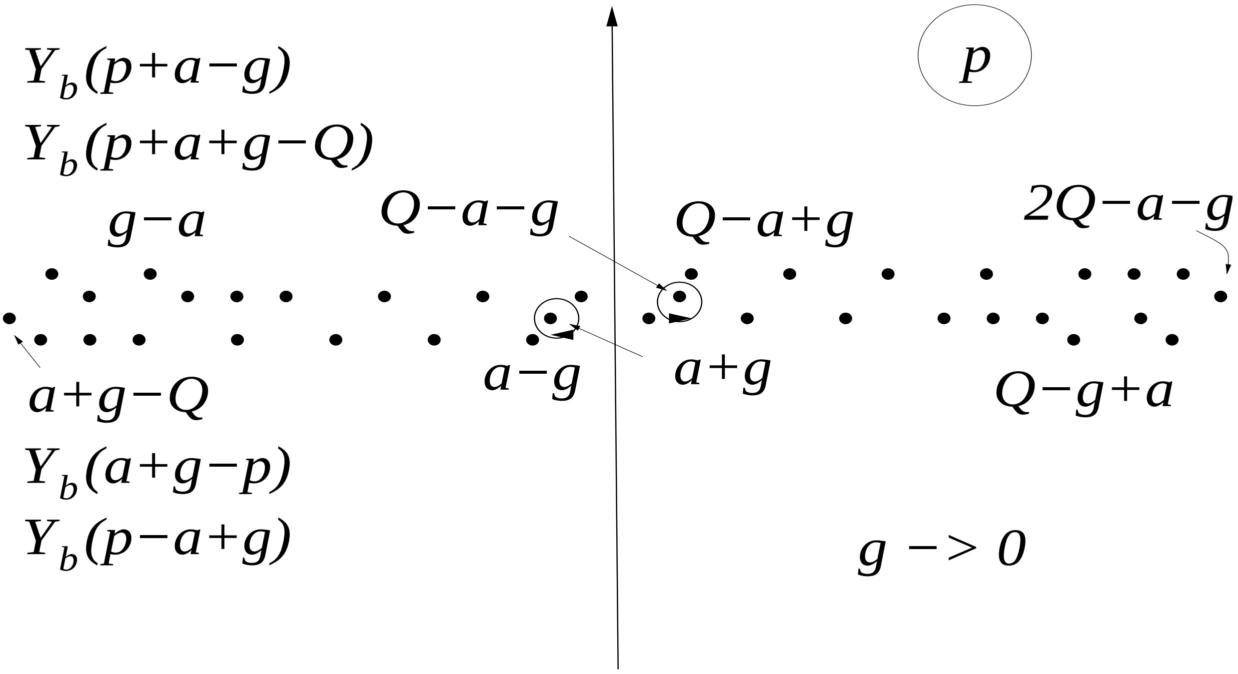
  $$V_g(y)V_a(0) = \frac{1}{2}(y\bar{y})^{-2ag}\left([V_{a+g}(0)] + R_{\mathrm{L}}(a+g)[V_{Q-a-g}(0)]\right) + \text{ integral term}$$

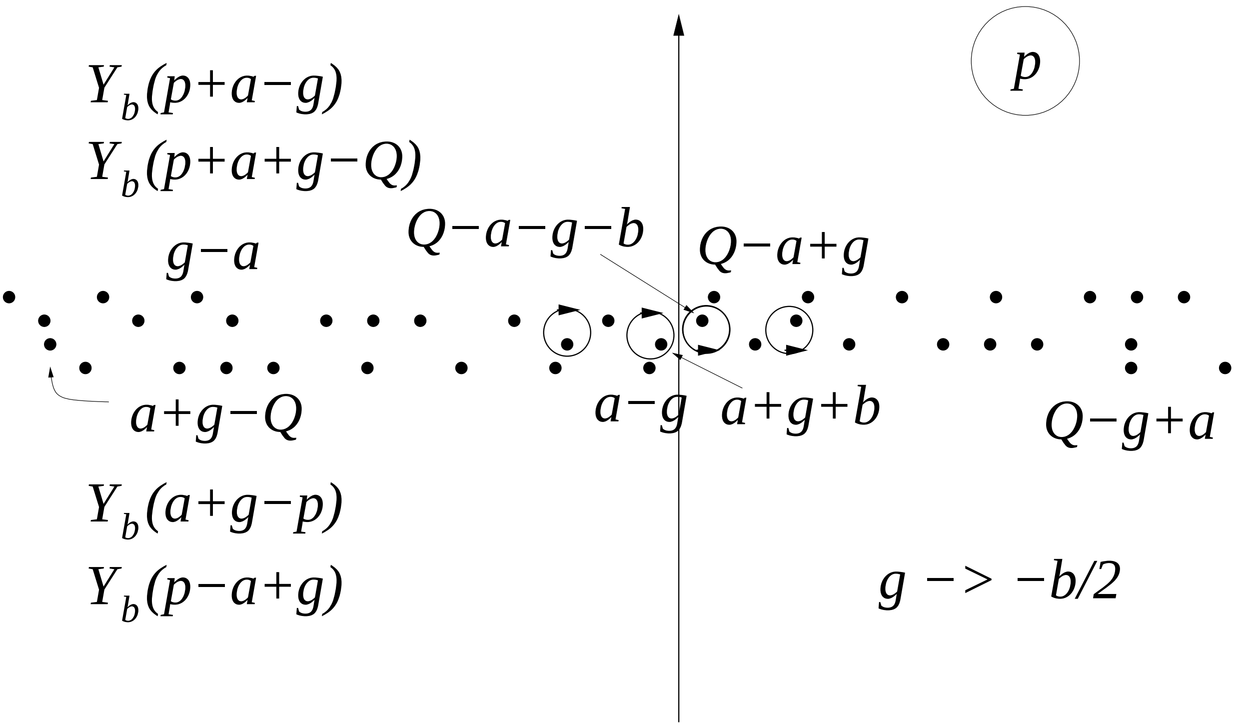


$Y_b(p+a-g)$
$Y_b(p+a+g-Q)$
$Q/2+iP$
p
$Q-a-g$
$g-a$
$Q-a+g$
$2Q-a-g$
$a+g-Q$
$a-g$
$a+g$
$Q-g+a$
$Y_b(a+g-p)$
$Y_b(p-a+g)$



$Y_b(p+a-g)$
$Y_b(p+a+g-Q)$
p
$Q-a-g$
$g-a$
$Q-a+g$
$2Q-a-g$
$a+g-Q$
$a-g$
$a+g$
$Q-g+a$
$Y_b(a+g-p)$
$Y_b(p-a+g)$


Yb(p+a−g)
Yb(p+a+g−Q)
p
g−a
Q−a−g
Q−a+g
2Q−a−g
a+g−Q
a−g
a+g
Q−g+a
Yb(a+g−p)
Yb(p−a+g)
g −> 0


Yb(p+a−g)
Yb(p+a+g−Q)
p
g−a
Q−a−g−b
Q−a+g
a+g−Q
a−g
a+g+b
Q−g+a
Yb(a+g−p)
Yb(p−a+g)
g −> −b/2

with Liouville "reflection amplitude"

$$R_{\rm L}(a) =$$

$$\left(\pi\mu\gamma(b^2)\right)^{(Q-2a)/b} \frac{b^{-2}\gamma(2ab-b^2)}{\gamma(2-2ab^{-1}-b^{-2})}$$

IMPORTANT: Above implies $\mathrm{Re}(a+g) < Q/2$. If $\mathrm{Re}(a+g) > Q/2$ we have to pick up instead the poles $p = Q-a+g$ and $p = a-g$

$$V_g(y)V_a(0) = \frac{1}{2} R_{\rm L}(a)(y\bar{y})^{-2g(Q-a)}$$

$$([V_{Q-a+g}(0)] + R_{\rm L}(a-g)[V_{a-g}(0)])$$

$$+ \text{ integral term}$$

Limit $g \to 0$ gives (in both cases) $V_a(0)$ as expected: the integral term vanishes.

- **Derivative** w.r.t g before the limit ($\mathrm{Re}\, a < Q/2$)

$$\phi(y)V_a(0) = -a\log(y\bar{y})V_a(0) + \text{const}$$

$$+ \text{ less singular terms}$$

simulates the free field OPE. Denote

$$|x|_{\rm Re} = \begin{cases} x & \text{if } \mathrm{Re}\, x > 0 \\ -x & \text{if } \mathrm{Re}\, x < 0 \end{cases}$$

then (unlike free field)

$$\phi(y)V_a(0) =$$

$$(|Q/2-a|_{\rm Re} - Q/2)\log(y\bar{y})V_a(0) + \ldots$$

- **Point** $g \to -b/2$ is treated similarly. There are two singular terms. Direct limit –> two terms of the standard discrete OPE

$$V_{1,2}(y)V_a(0) = C_+^{(\mathrm{L})}(a)(y\bar{y})^{ab}\left[V_{a-b/2}\right] + C_-^{(\mathrm{L})}(a)(y\bar{y})^{1-ab+b^2}\left[V_{a+b/2}\right]$$

with correct structure constants

$$C_+^{(\mathrm{L})}(a) = 1$$
$$C_-^{(\mathrm{L})}(a) = -\frac{\pi\mu}{\gamma(-b^2)}\frac{\gamma(2ab-b^2-1)}{\gamma(2ab)}$$

For the logarithmic field

$$V'_{1,2}(y)V_a(0) = \log(y\bar{y})\times \Big(q_{0,1}^{(1,2)}(a)(y\bar{y})^{ab}C_+^{(\mathrm{L})}(a)V_{a-b/2}(0)+ q_{0,-1}^{(1,2)}(a)(y\bar{y})^{1-ab+b^2}C_-^{(\mathrm{L})}(a)V_{a+b/2}(0)\Big) + \dots$$

where

$$q_{0,s}^{(1,2)}(a) = |a - bs/2 - Q/2|_{\mathrm{Re}} - \lambda_{1,2}$$

- **OPE** $O'_{1,2}(x)\mathcal{W}_a$ goes as for the discrete states – “wrong terms” go away due to $H_{1,2}$ while "good" terms combine to

$$O'_{1,2}(x)\mathcal{W}_a = \Lambda_{1,2}\log(x\bar{x})\times \Big(q^{(1,2)}_{0,1}(a)\mathcal{W}_{a-b/2} + q^{(1,2)}_{0,-1}(a)\mathcal{W}_{a+b/2}\Big) + \text{ less singular terms}$$

- **General logarithmic OPE** $O'_{m,n}(x)\mathcal{W}_a$

$$\mathcal{O}'_{m,n}(x)\mathcal{W}_a = \log(x\bar{x}) \sum_{r,s}^{(m,n)} q^{(m,n)}_{r,s}(a)\mathcal{W}_{a-\lambda_{r,s}}$$

with the sum over the standard for (m,n) set of (r,s)

$$q^{(m,n)}_{r,s}(a) = |a-\lambda_{r,s}-Q/2|_{\mathrm{Re}} - \lambda_{m,n}$$

and

$$\mathcal{O}'_{m,n}(x) = \Lambda^{-1}_{m,n}O'_{m,n}$$

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- **Sums up to** (together with curvature term)

$$Z_{\mathrm{L}}^{-1}\int \langle \mathcal{U}_{m,n}(x)\mathcal{W}_{a_1}\mathcal{W}_{a_2}\mathcal{W}_{a_3}\rangle\, d^2x = \pi^4(b^{-2}+1)b^{-3}(b^{-2}-1)\Sigma_{m,n}(a_1,a_2,a_3)$$

with

$$\Sigma_{m,n}(a_1,a_2,a_3) = -2mn\lambda_{m,n}$$
$$-\sum_{i=1}^{3}\sum_{r,s}^{(m,n)}\left(|a_i-\lambda_{r,s}-Q/2|_{\mathrm{Re}}-\lambda_{m,n}\right)$$
$$= -\sum_{i=1}^{3}\sum_{r,s}^{(m,n)}|a_i-\lambda_{r,s}-Q/2|_{\mathrm{Re}}+mn\lambda_{m,n}$$

10 Comparing with matrix models

- **Critical** $O(n)$-model, or critical loop gaz with

$$n = 2\cos\pi(g-1)$$

is described by GMM $\mathcal{M}_{b^2}$ with $1 < g < 2$ $(g = b^{-2})$

$$c_{\mathrm{M}} = 13 - 6(g + g^{-1})$$

- **Off-criticality:** “massive” loops of mass t. Spherical partition function $Z(x,t)$ (I.Kostov, 2005)

$$u = -(g-1)Z_{xx}$$

– $x \sim$ cosmological constant, $p = 1/(g-1)$

$$u^p + tu^{p-1} = x$$

- **Perturbative expansion** in t

$$Z(x,t) = x^{g+1} \sum_{n=0}^{\infty} \frac{\Gamma((g-1)n-g-1)}{n!\Gamma((g-2)n-g+2)} (tx^{1-g})^n$$

- **Correlation functions** of $\mathcal{U}_{1,3} = \mathcal{U}$ are read off

$$\begin{aligned} \langle\langle \mathcal{U}^2 \rangle\rangle &= -\frac{(g+1)g(g-1)}{g-3} \\ \langle\langle \mathcal{U}^3 \rangle\rangle &= -(g+1)g(g-1) \\ \langle\langle \mathcal{U}^4 \rangle\rangle &= -3(g+1)g(g-1)(g-2) \end{aligned}$$

Our expression:

$$(g+1)g(g-1) \times b^{-1}\Sigma$$

where

$$\begin{aligned} b^{-1}\Sigma &= \frac{3}{2}\left(g+3-|g-1|-|g-3|-|g-5|\right) \\ &= -3(g-2) \end{aligned}$$