Topological Insulators

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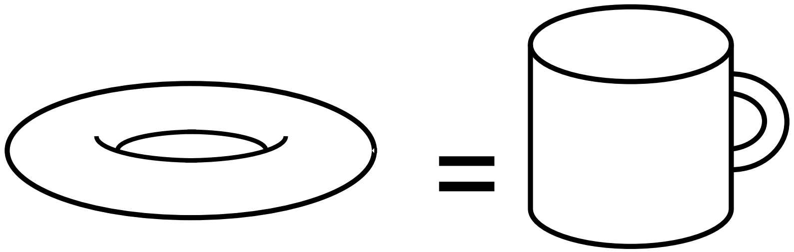
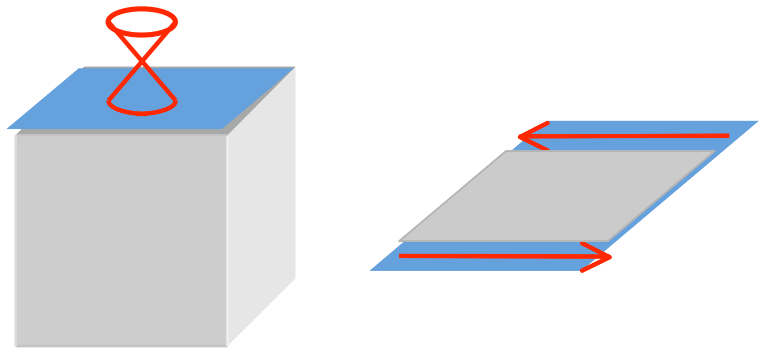




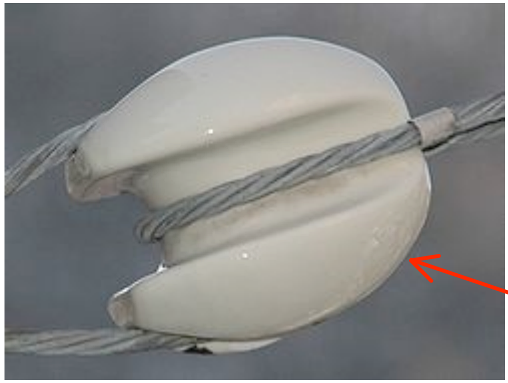
topological insulators (3d and 2d)

Outline

- Introduction: band theory
- Example of topological insulators:
  integer quantum Hall effect
- New members: Z_2 topological insulators
- Table of topological insulators

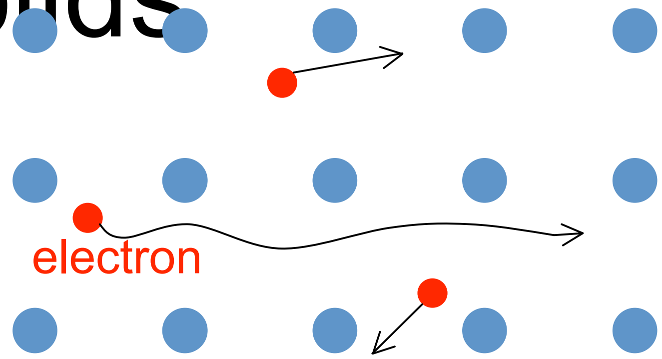
Insulator

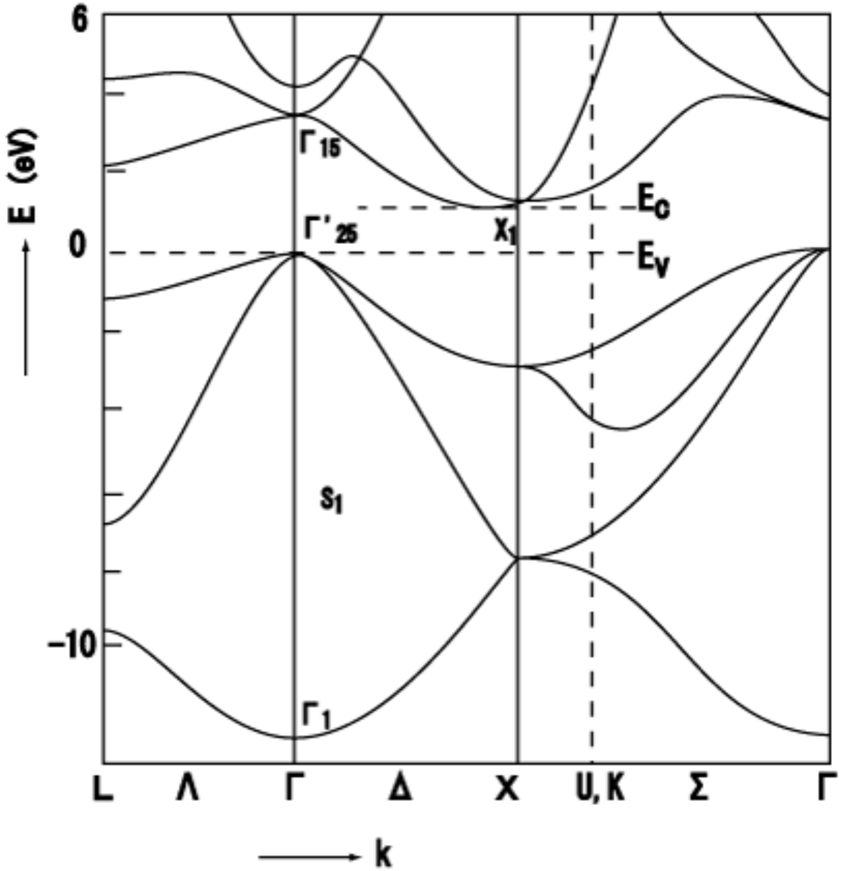
- A material which resists the flow of electric current.



insulating materials

Band theory of electrons in solids





- Schroedinger equation

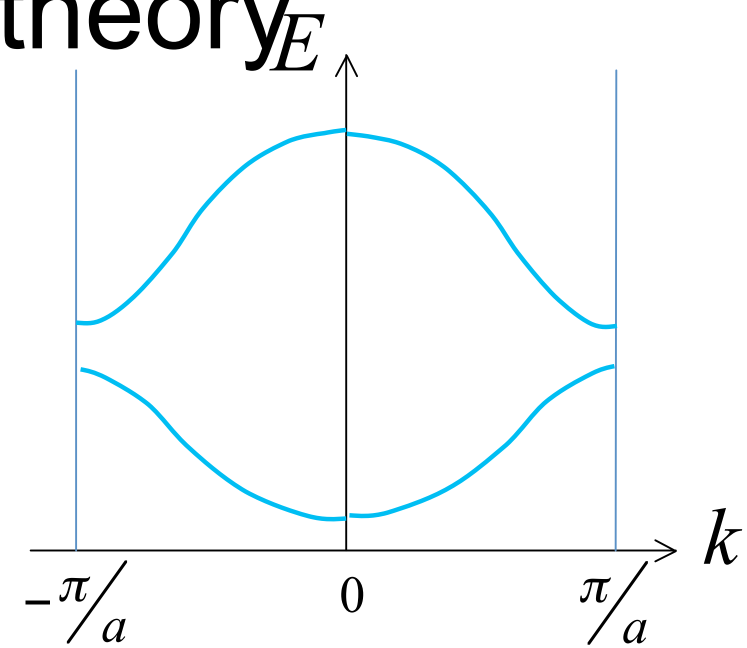
$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(r) = E\psi(r)$$

Bloch's theorem:

$$\psi(r) = e^{ikr}u_{n,k}(r), \quad -\frac{\pi}{a} < k < \frac{\pi}{a}, \quad u_{n,k}(r+a) = u_{n,k}(r)$$

$E_n(k)$ Energy band dispersion $\qquad$ n : band index

Metal and insulator in the band theory

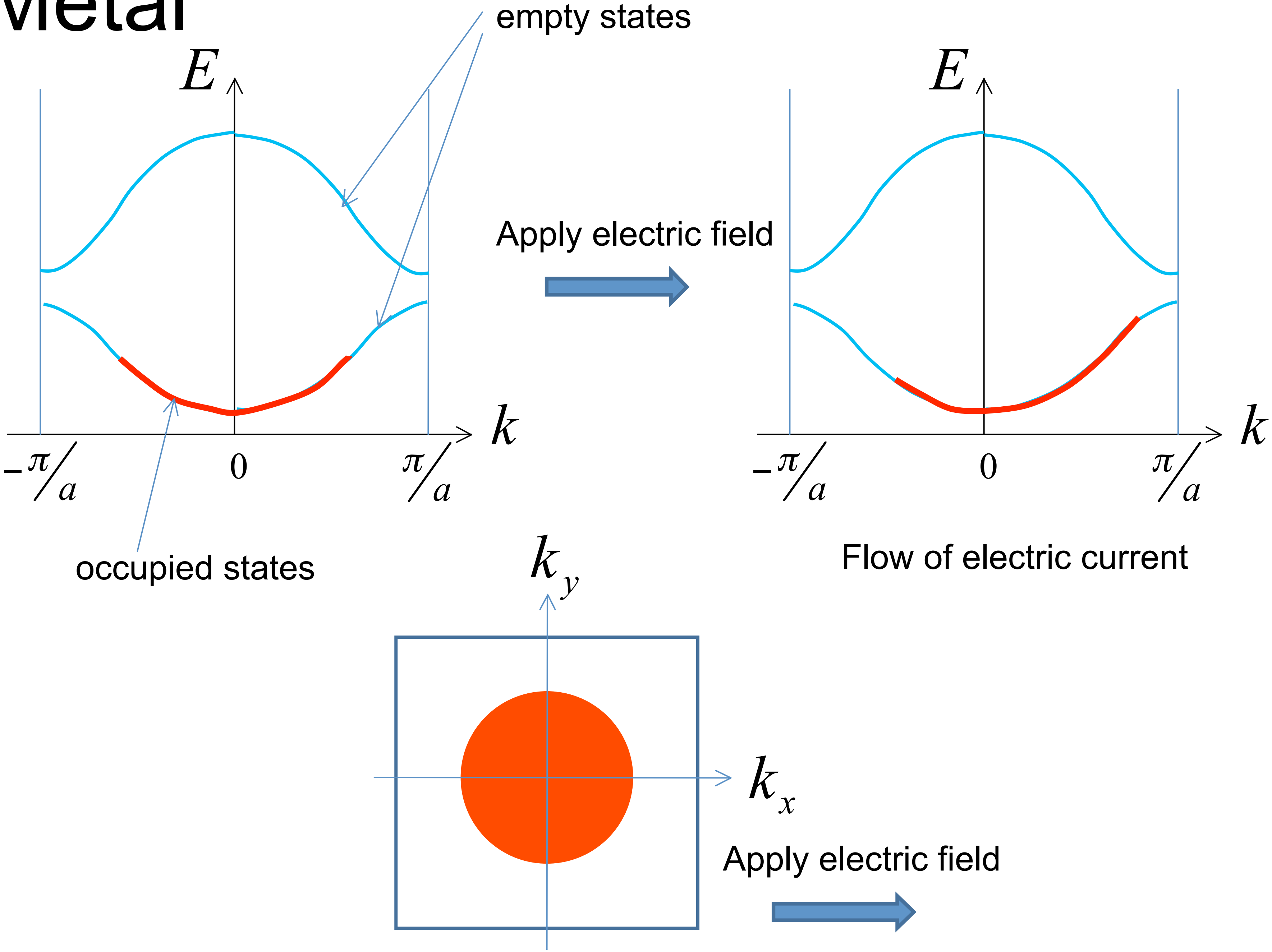


Electrons are fermions (spin=1/2).

Each state (n,k) can accommodate up to two electrons (up, down spins).

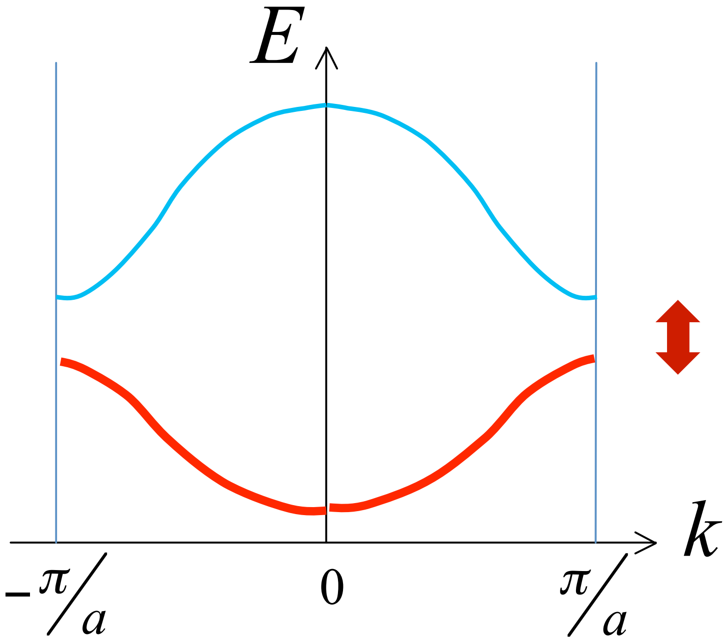
Pauli principle

Metal



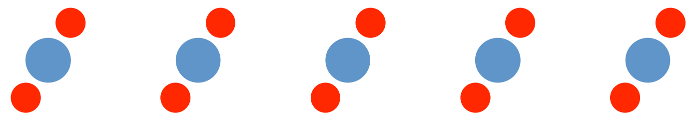
empty states
E
k
−π/a
0
π/a
occupied states
Apply electric field
E
k
−π/a
0
π/a
Flow of electric current
k_y
k_x
Apply electric field

Band Insulator



Band gap $\left|\tilde{V}(k=\pi/a)\right|$

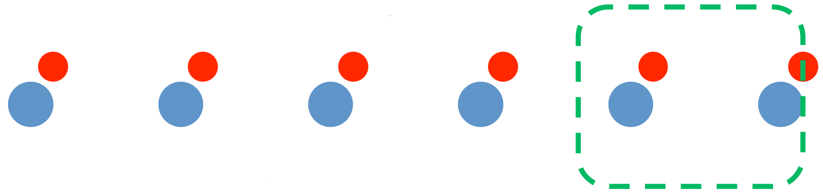
All the states in the lower band are completely filled.
(2 electrons per unit cell)

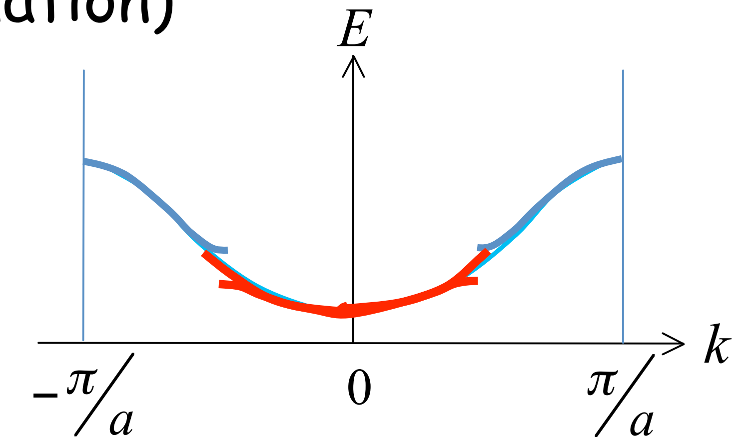


Electric current does not flow under (weak) electric field.

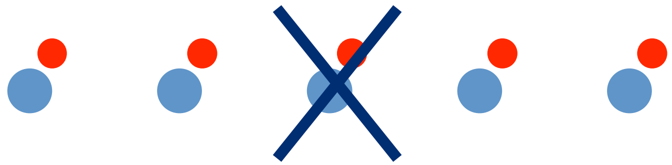
Digression: other (named) insulators

- Peierls insulator (lattice deformation)



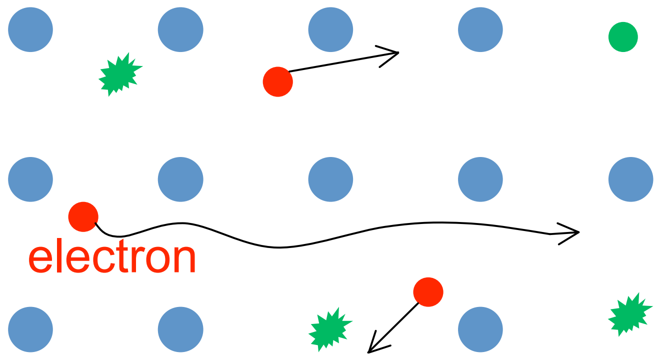


- Mott insulator (Coulomb repulsion)



Large Coulomb energy! Electrons cannot move.

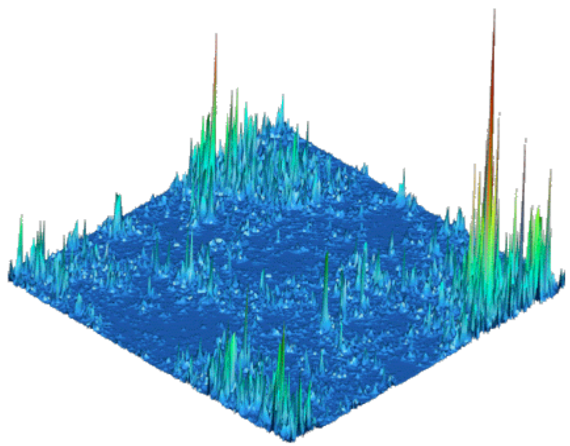
- Anderson insulator (impurity scattering)



Random scattering causes interference of electron's wave function.

→ standing wave

Anderson localization



Is that all?

No !

Yet another type of insulators: Topological insulators !

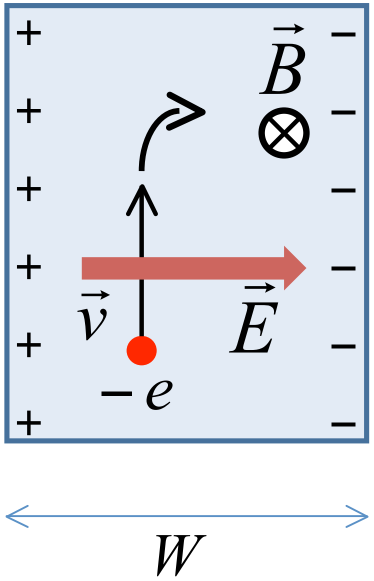
A topological insulator is a band insulator

which is characterized by a topological number and

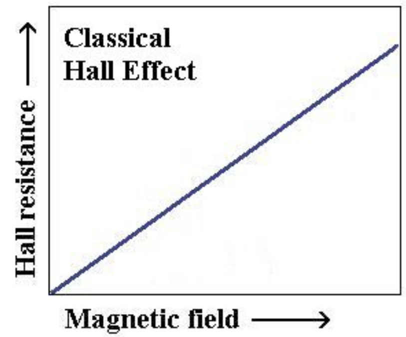
which has gapless excitations at its boundaries.

Prominent example: quantum Hall effect

- Classical Hall effect



n : electron density



Electric current $I = -nevW$

Electric field $E = \frac{v}{c}B$

Hall voltage $V_H = EW = \frac{B}{-ne}I$

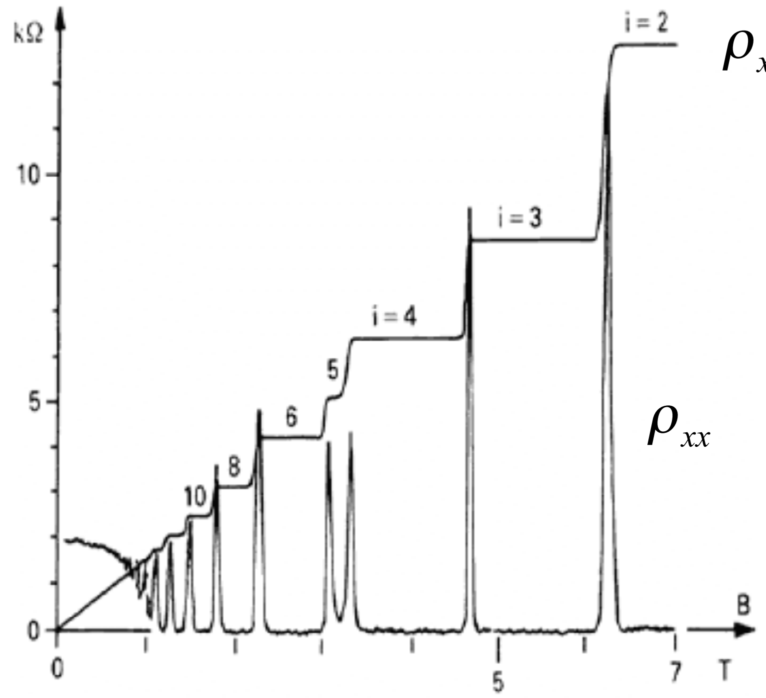
Hall resistance $R_H = \frac{B}{-ne}$

Hall conductance $\sigma_{xy} = \frac{1}{R_H}$

Lorentz force

$\vec{F} = -e\vec{v} \times \vec{B}$

Integer quantum Hall effect (von Klitzing 1980)



$$\frac{h}{e^2} = 25812.807\Omega$$

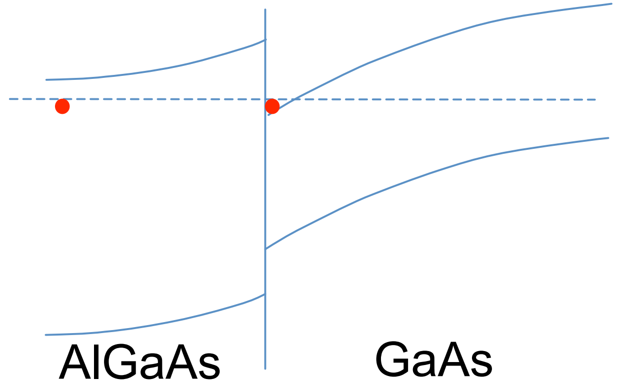
Quantization of Hall conductance

$$\sigma_{xy} = i\frac{e^2}{h}$$

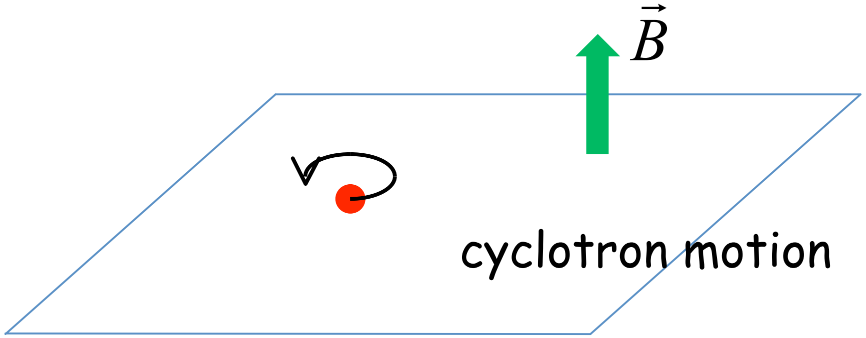
exact, robust against disorder etc.

Integer quantum Hall effect

- Electrons are confined in a two-dimensional plane. (ex. AlGaAs/GaAs interface)

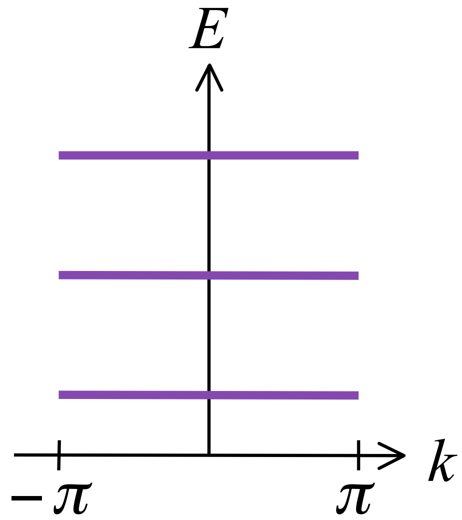


- Strong magnetic field is applied (perpendicular to the plane)



Landau levels:

$$E_n = \hbar\omega_c\left(n + \tfrac{1}{2}\right), \quad \omega_c = \frac{eB}{mc}, \quad n = 0,1,2,\ldots$$



TKNN number (Thouless-Kohmoto-Nightingale-den Nijs)

TKNN (1982); Kohmoto (1985)

$$\sigma_{xy} = -\frac{e^2}{h} C$$

first Chern number (topological invariant)

$$\psi = e^{i\vec{k}\cdot\vec{r}} u_{\vec{k}}(\vec{r})$$

$$C = \frac{1}{2\pi i} \int_{\text{filled band}} d^2k \int d^2r \left(\frac{\partial u^*}{\partial k_y} \frac{\partial u}{\partial k_x} - \frac{\partial u^*}{\partial k_x} \frac{\partial u}{\partial k_y} \right)$$

integer valued

$$= \frac{1}{2\pi i} \int d^2k \, \vec{\nabla}_k \times \vec{A}(k_x, k_y)$$

$$\vec{A}(k_x, k_y) = \langle u_{\vec{k}} | \vec{\nabla}_k | u_{\vec{k}} \rangle$$

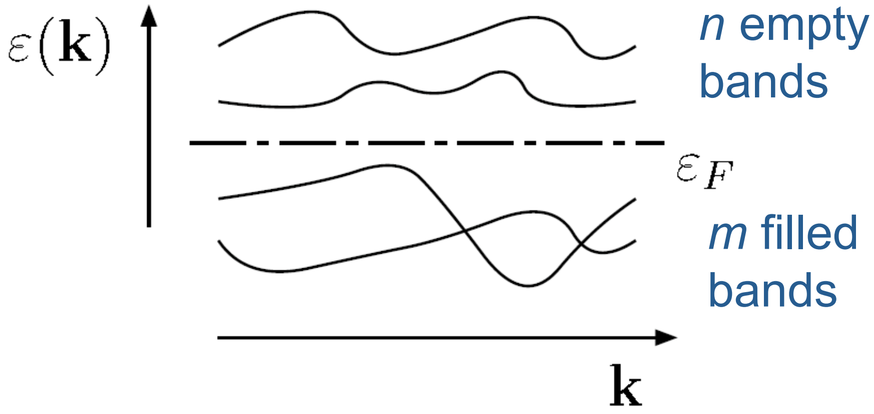
Topological distinction of ground states

projection operator

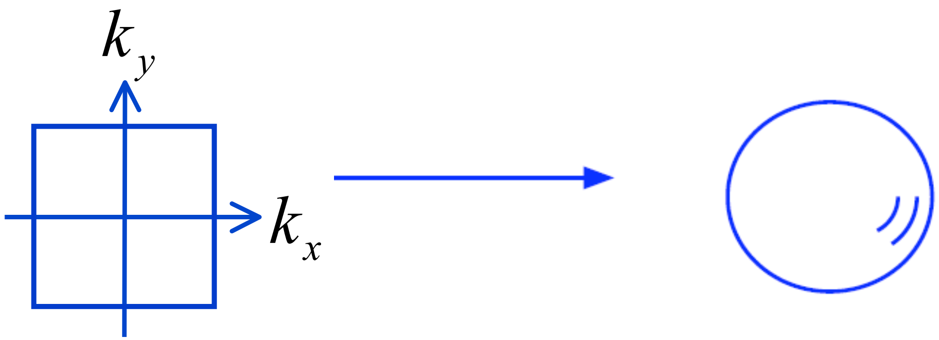
$$Q(k) = 2 \sum_{a \in \text{filled}} |u_a(k)\rangle\langle u_a(k)| - 1$$

$$Q^2 = 1, \; Q^\dagger = Q, \; \mathrm{tr}\, Q = m - n$$

filled (→ m) empty (→ n)



$$Q \; : \text{BZ} \longrightarrow U(m+n)/U(m)\times U(n)$$



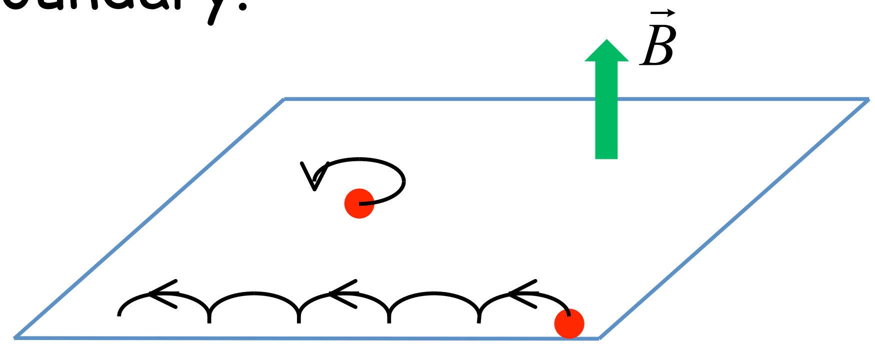
map from BZ to Grassmannian

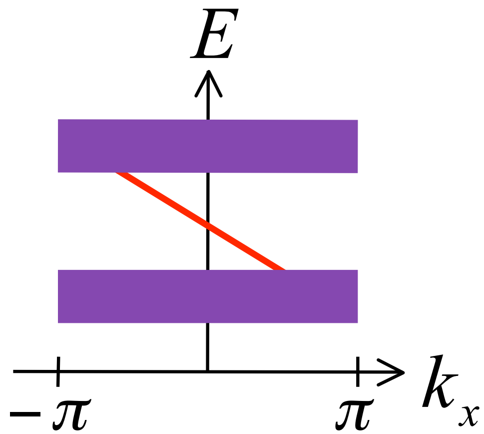
$$\pi_2\left[U(m+n)/U(m)\times U(n)\right] = Z \quad \Longrightarrow \quad \text{IQHE}$$

homotopy class

Edge states

- There is a gapless chiral edge mode along the sample boundary.



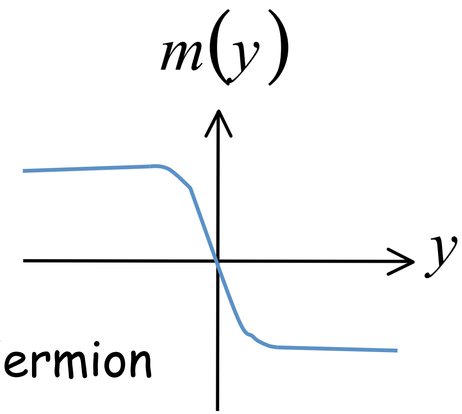


$$\text{Number of edge modes} = \frac{-\sigma_{xy}}{e^2/h} = C$$

Robust against disorder (chiral fermions cannot be backscattered)

Effective field theory

$$H = -iv\left(\sigma_x \partial_x + \sigma_y \partial_y\right) + m(y)\sigma_z$$



domain wall fermion

Topological Insulators (definition ??)

- (band) insulator with a nonzero gap to excitated states
- topological number

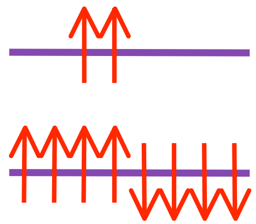
  stable against any (weak) perturbation
- gapless edge mode

- When the gapless mode appears/disappears, the bulk (band) gap closes. Quantum Phase Transition

- Low-energy effective theory

  = topological field theory (Chern-Simons)

Fractional quantum Hall effect at $\nu = \frac{5}{2}$

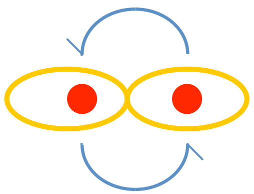


- 2nd Landau level
- Even denominator (cf. Laughlin states: odd denominator)
- Moore-Read (Pfaffian) state

$$z_j = x_j + iy_j$$

$$\psi_{\mathrm{MR}} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^2 e^{-\sum |z_i|^2} \qquad \mathrm{Pf}(A_{ij}) = \sqrt{\det A_{ij}}$$

Pf( ) is equal to the BCS wave function of $p_x + ip_y$ pairing state.



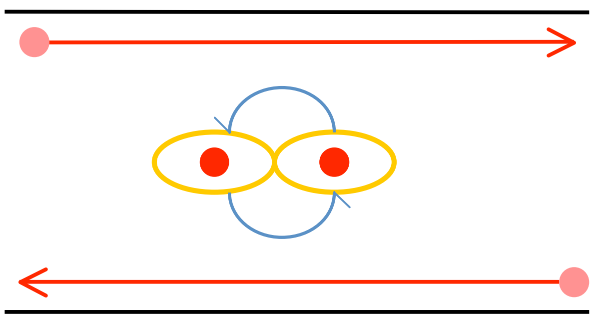
Bound state of two spinless ferions:
P-wave & angular momentum=1

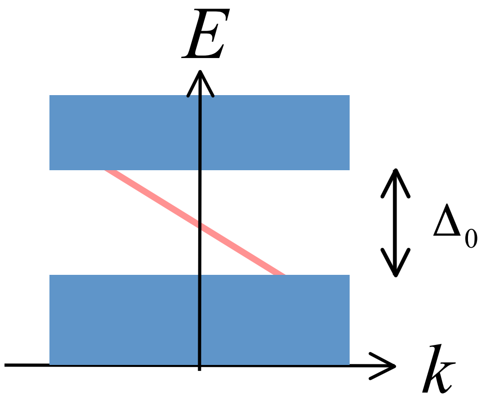
Excitations above the Moore-Read state obey non-Abelian statistics.

Spinless p_x+ip_y superconductor in 2 dim.

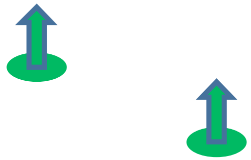
- Order parameter $\Delta(\vec{k}) \propto \langle \Psi_k \Psi_{-k} \rangle \propto \Delta_0 (k_x + ik_y)$

- Chiral (Majorana) edge state





- Majorana bound state in a quantum vortex



vortex $\phi = \frac{hc}{e}$

Bogoliubov-de Gennes equation

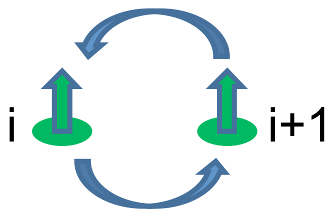
$$\begin{pmatrix} h_0 & \Delta \\ \Delta^* & -h_0^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \varepsilon \begin{pmatrix} u \\ v \end{pmatrix} \qquad h_0 = \frac{1}{2m}\left(\vec{p} + e\vec{A}\right)^2 - E_F \qquad \begin{pmatrix} \Psi \\ \Psi^+ \end{pmatrix} \Leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix}$$

particle-hole symmetry $\varepsilon : \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow -\varepsilon : \begin{pmatrix} v^* \\ u^* \end{pmatrix}$

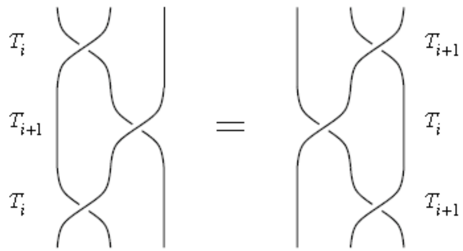
$$\varepsilon_n = n\omega_0, \quad \omega_0 \approx \Delta_0^2 / E_F$$

zero mode $\varepsilon_0 = 0$ $\quad \Psi = \Psi^+ \; (=\gamma)$ Majorana (real) fermion!

interchanging vortices ⇒ braid groups, non-Abelian statistics



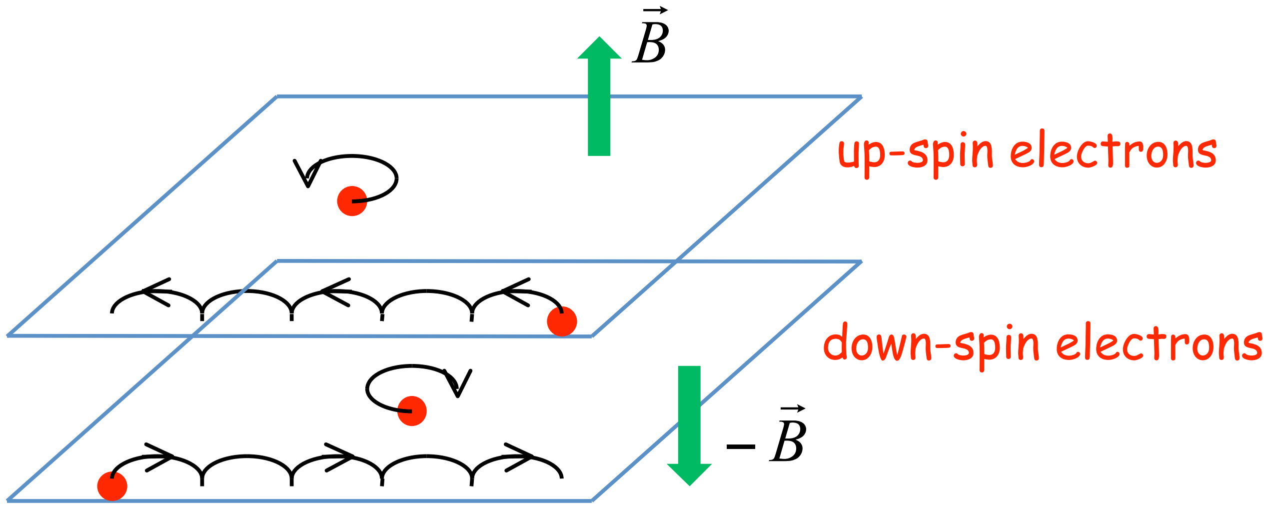
$$\gamma_i \rightarrow \gamma_{i+1}$$
$$\gamma_{i+1} \rightarrow -\gamma_i$$



Quantum spin Hall effect (Z_2 top. Insulator)

Kane & Mele (2005, 2006); Bernevig & Zhang (2006)

- Time-reversal invariant band insulator
- Strong spin-orbit interaction $\lambda \vec{L} \cdot \vec{\sigma}$
- Gapless helical edge mode (Kramers pair)

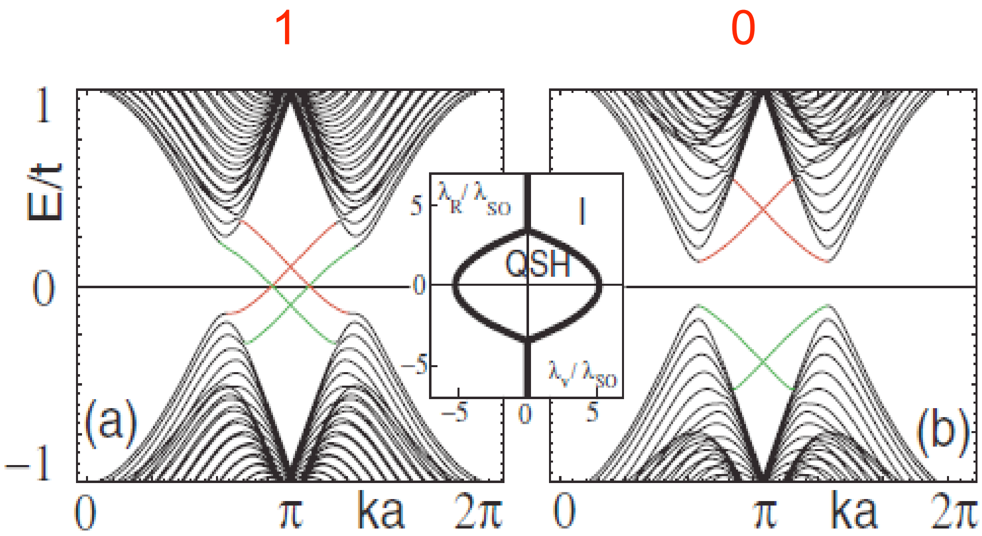


No spin rotation symmetry

- Quantum spin Hall insulator is characterized by Z_2 topological index ν

$\nu = 1$ an odd number of helical edge modes; Z_2 topological insulator

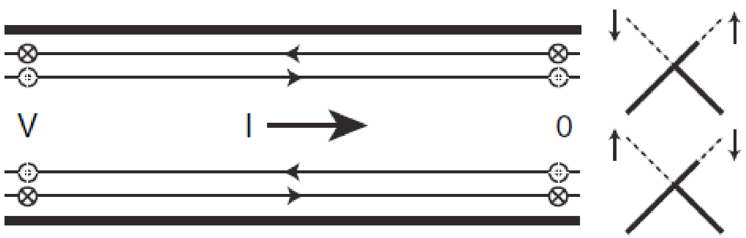
$\nu = 0$ an even (0) number of helical edge modes



# of a pair of zeros of

$$\mathrm{Pf}\left[\left\langle u_i(\vec{k})\middle| i s_y \middle| u_j(\vec{k})\right\rangle^*\right]$$

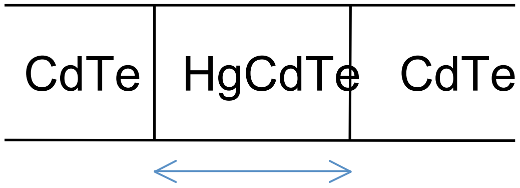
Kane-Mele model
graphene + SOI
[PRL 95, 146802 (2005)]

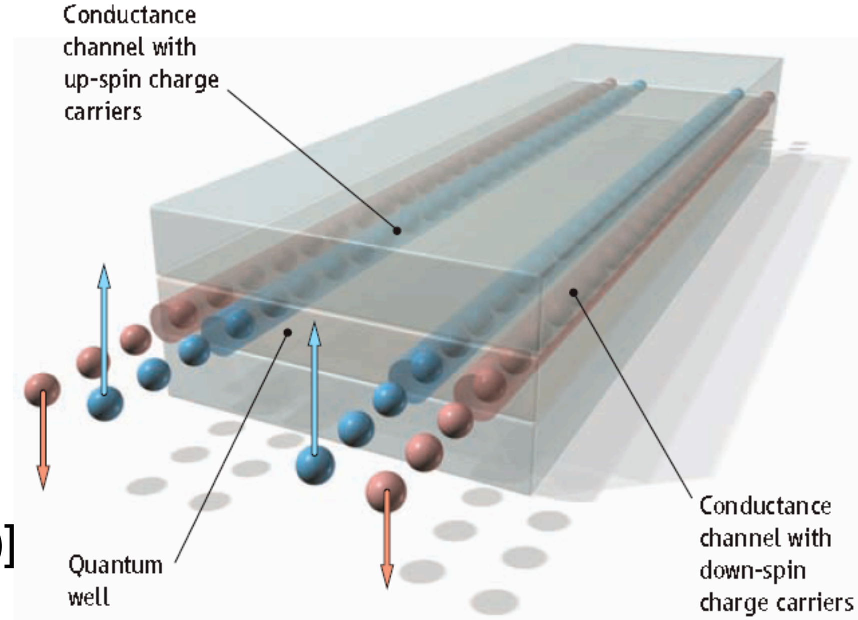


Quantum spin Hall effect $\sigma^s_{xy} = \dfrac{e}{2\pi}$

Experiment

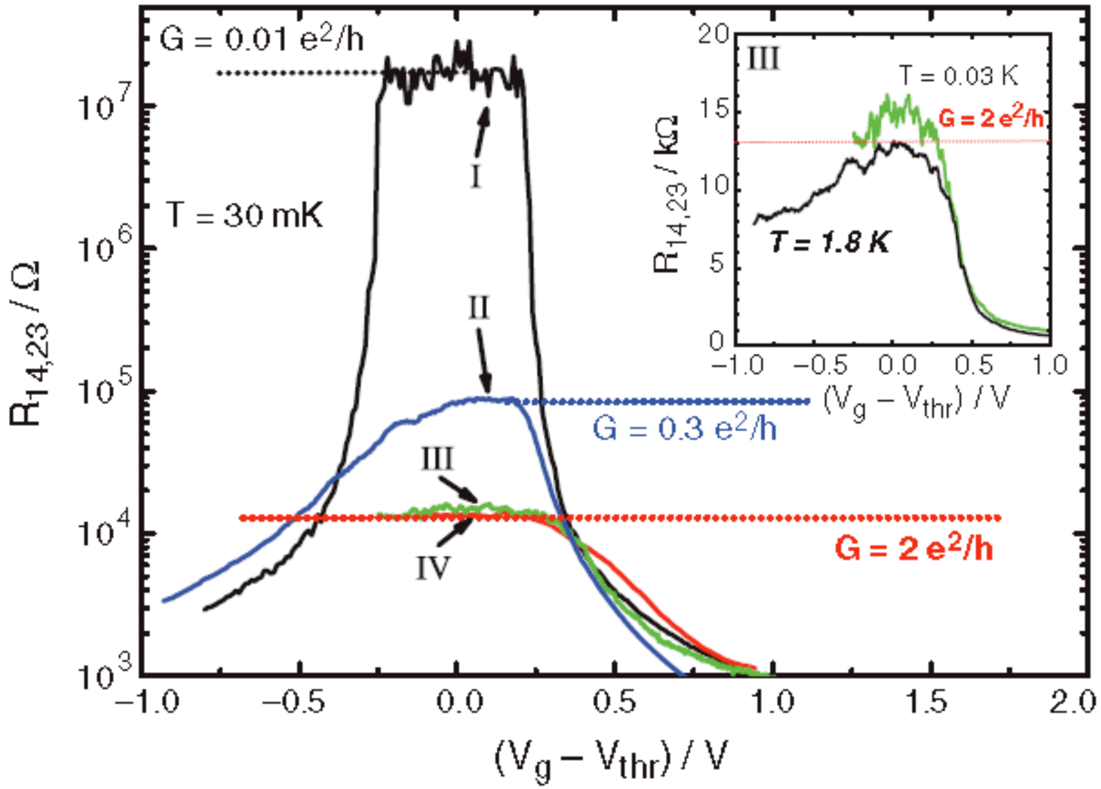
HgTe/(Hg,Cd)Te quantum wells





Konig et al. [Science 318, 766 (2007)]

Fig. 4. The longitudinal four-terminal resistance, $R_{14,23}$, of various normal (d = 5.5 nm) (I) and inverted (d = 7.3 nm) (II, III, and IV) QW structures as a function of the gate voltage measured for B = 0 T at T = 30 mK. The device sizes are (20.0 × 13.3) μm^2 for devices I and II, (1.0 × 1.0) μm^2 for device III, and (1.0 × 0.5) μm^2 for device IV. The inset shows $R_{14,23}(V_g)$ of two samples from the same wafer, having the same device size (III) at 30 mK (green) and 1.8 K (black) on a linear scale.



3-dimensional Z_2 topological insulator

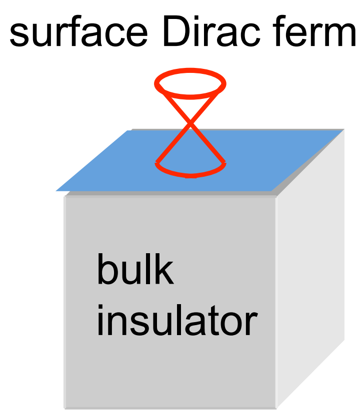
Moore & Balents; Roy; Fu, Kane & Mele (2006, 2007)

(strong) topological insulator

bulk: band insulator

surface: an odd number of surface Dirac modes

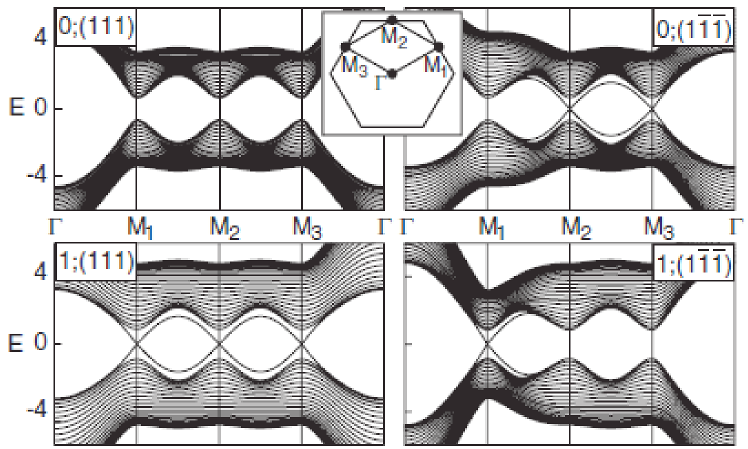
characterized by Z_2 topological numbers



Ex: tight-binding model with SO int. on the diamond lattice
[Fu, Kane, & Mele; PRL 98, 106803 (2007)]

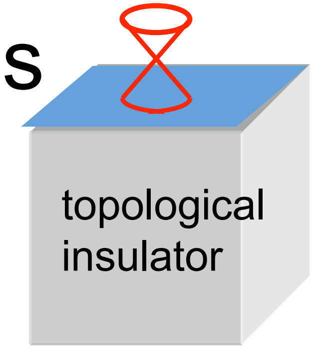
trivial insulator

Z_2 topological insulator

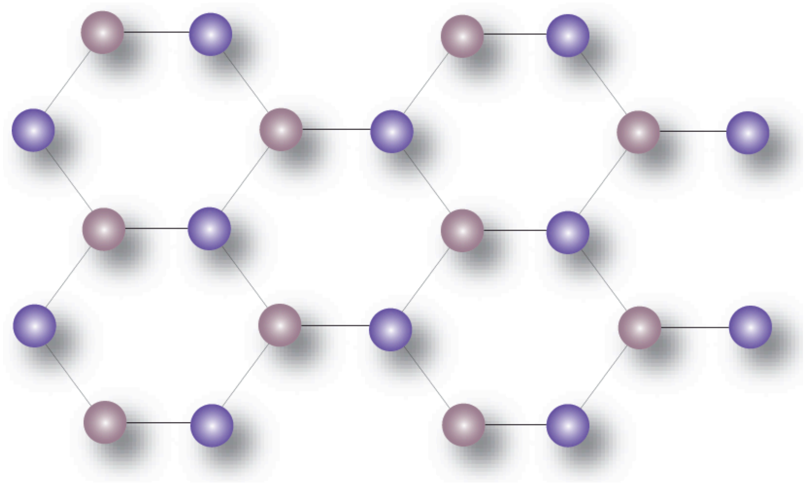


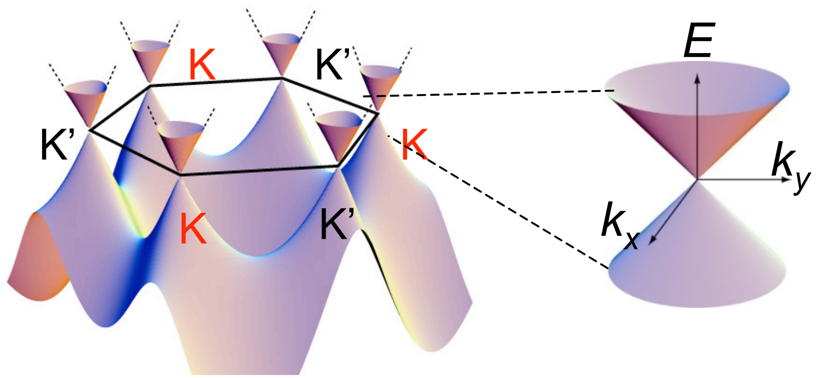
trivial band insulator:
0 or an even number of surface Dirac modes

Surface Dirac fermions



- A "half" of graphene

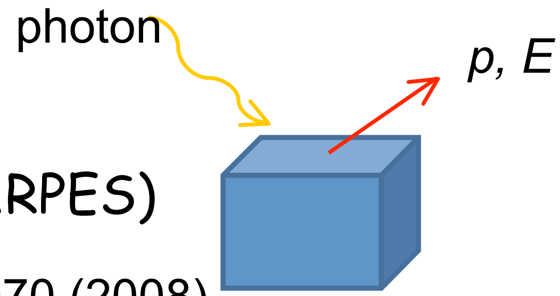




- An odd number of Dirac fermions in 2 dimensions

  cf. Nielsen-Ninomiya's no-go theorem

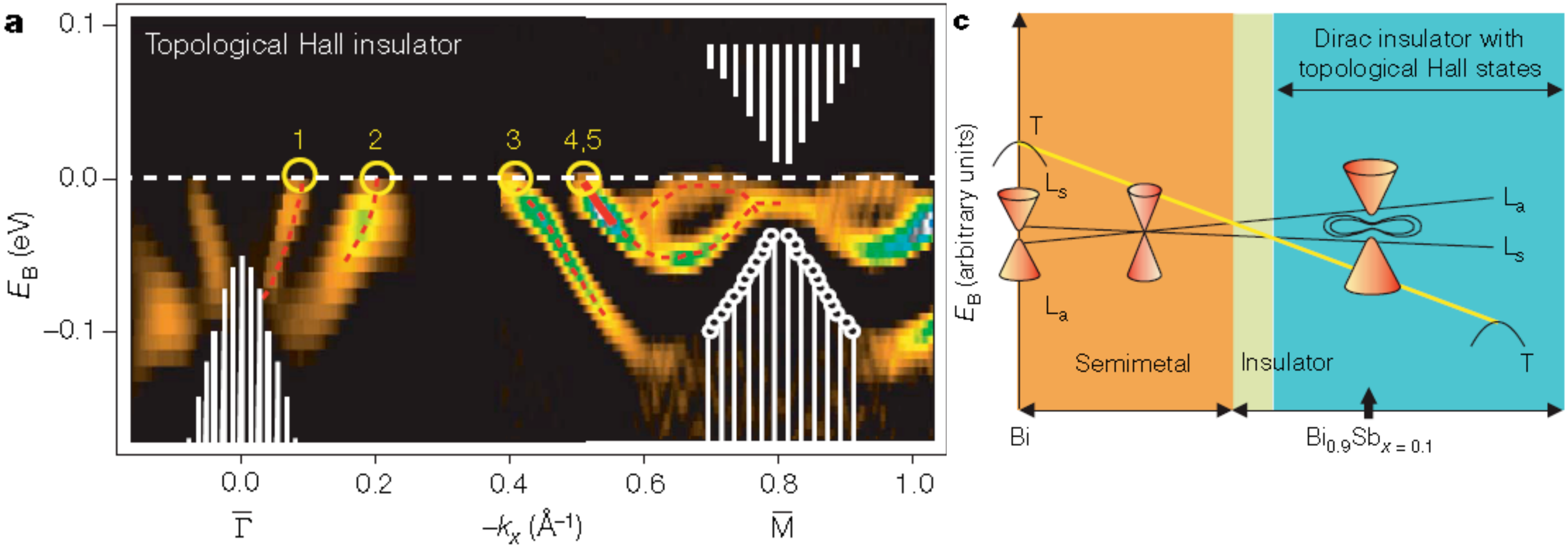
Experiments



- *Angle-resolved photoemission spectroscopy (ARPES)*

$Bi_{1-x}Sb_x$

Hsieh et al., Nature 452, 970 (2008)



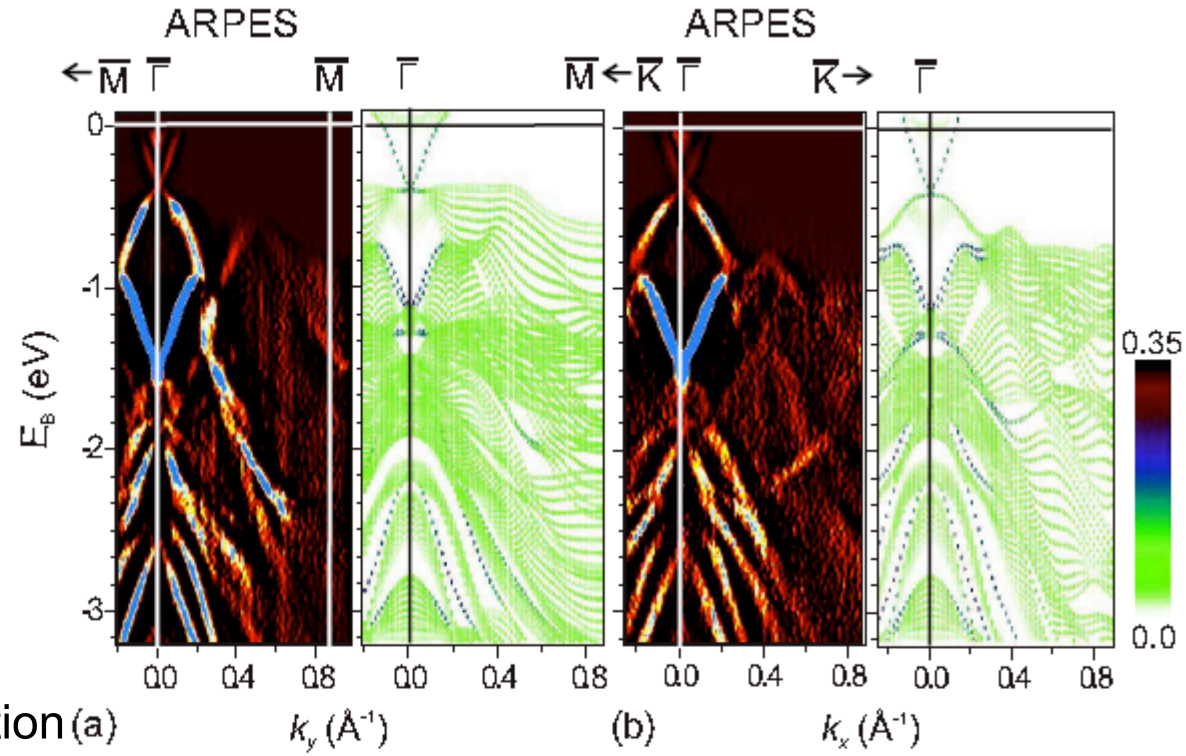
An odd (5) number of surface Dirac modes were observed.

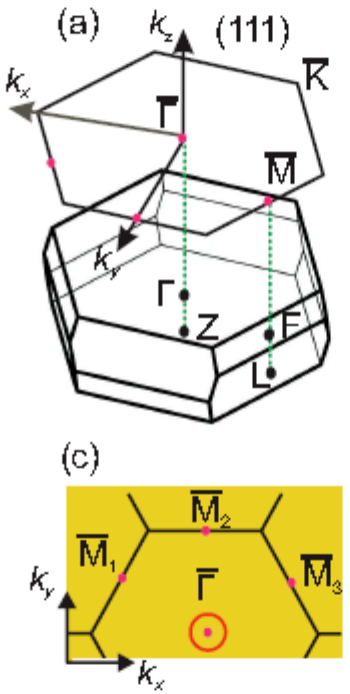
Experiments II

Bi_2Se_3

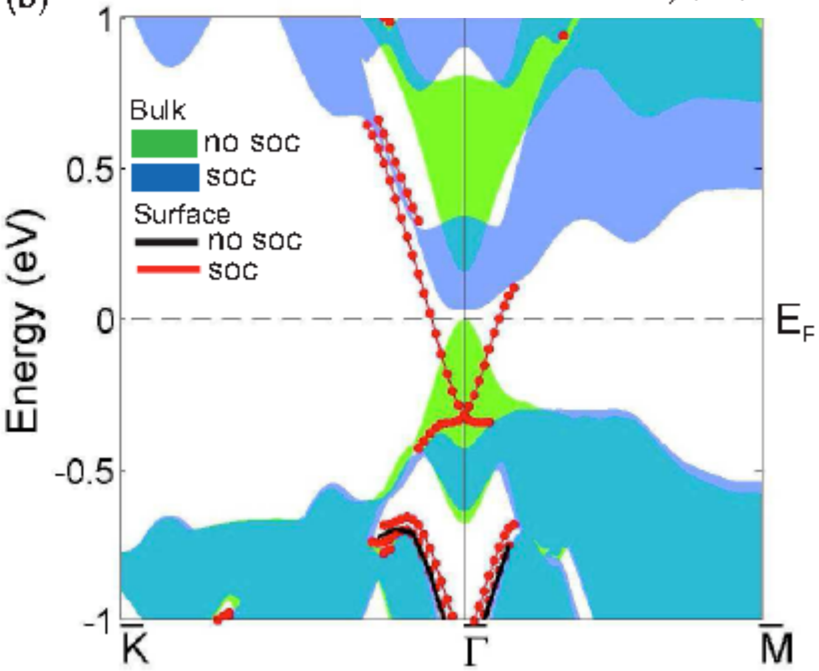
"hydrogen atom" of top. ins.

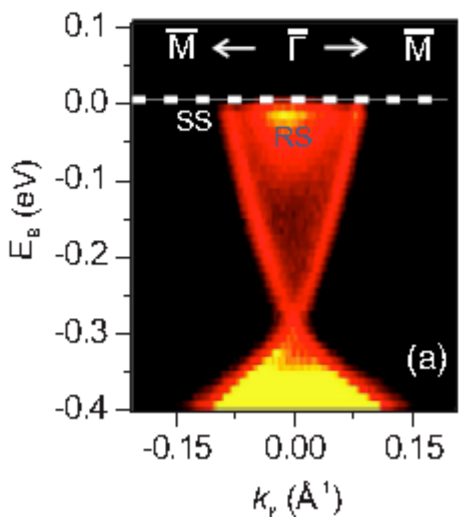
Xia et al., arXiv:0812.2078





band calculation





Q: Are there other 3D topological insulators?

Yes!

Let's make a table of all possible topological insulators.

Classification of topological insulators

Topological insulators are stable against (weak) perturbations.

Random deformation of Hamiltonian

Natural framework: random matrix theory
(Wigner, Dyson, Altland & Zirnbauer)

Assume only basic discrete symmetries:

(1) time-reversal symmetry

$$TH^{*}T^{-1} = H$$

TRS =
- 0 no TRS
- +1 TRS with $T^{\mathrm{T}} = +T$ (integer spin)
- -1 TRS with $T^{\mathrm{T}} = -T$ (hal-odd integer spin

(2) particle-hole symmetry

$$CH^{\mathrm{T}}C^{-1} = -H$$

PHS =
- 0 no PHS
- +1 PHS with $C^{\mathrm{T}} = +C$ (odd parity: p-wave)
- -1 PHS with $C^{\mathrm{T}} = -C$ (even parity: s-wave

(3) TRS × PHS = chiral symmetry [sublattice symmetry (SLS)]

$$TCH(TC)^{-1} = -H$$

$$3 \times 3 + 1 = 10$$

10 random matrix ensembles

		TRS	PHS	SLS	description
Wigner-Dyson (standard)	A	0	0	0	unitary IQHE
	AI	+1	0	0	orthogonal
	AII	-1	0	0	symplectic (spin-orbit) Z_2 TPI
chiral (sublattice)	AIII	0	0	1	chiral unitary
	BDI	+1	+1	1	chiral orthogonal
	CII	-1	-1	1	chiral symplectic
BdG	D	0	+1	0	singlet/triplet SC MR Pfaffian
	C	0	-1	0	singlet SC
	DIII	-1	+1	1	singlet/triplet SC with TRS
	CI	+1	-1	1	singlet SC with TRS

Examples of topological insulators in 2 spatial dimensions

Integer quantum Hall Effect

Z_2 topological insulator (quantum spin Hall effect) also in 3D

Moore-Read Pfaffian state (spinless p+ip superconductor)

Table of topologocal insulators in 1, 2, 3 dim.

Schnyder, Ryu, Furusaki & Ludwig, PRB (200

random matrix ensemble		TRS	PHS	chS		d=1	d=2	d=3
Wigner-Dyson (standard)	A	0	0	0	unitary	0	$\mathbb{Z}$ [(a)]	0
	AI	+1	0	0	orthogonal	0	0	0
	AII	−1	0	0	symplectic	0	$\mathbb{Z}_2$ [(b)]	$\mathbb{Z}_2$ [(c)]
Chiral (sublattice)	AIII	0	0	1	chiral unitary	$\mathbb{Z}$	0	$\mathbb{Z}$
	BDI	+1	+1	1	chiral orthogonal	$\mathbb{Z}$	0	0
	CII	−1	−1	1	chiral symplectic	$\mathbb{Z}$	0	$\mathbb{Z}_2$
BdG	D	0	+1	0	(triplet) SC	$\mathbb{Z}_2$	$\mathbb{Z}$ [(d)]	0
	C	0	−1	0	singlet SC	0	$\mathbb{Z}$ [(e)]	0
	DIII	−1	+1	1	triplet SC	$\mathbb{Z}_2$	$\mathbb{Z}_2$ [(f)]	$\mathbb{Z}$ [(g)]
	CI	+1	−1	1	singlet SC	0	0	$\mathbb{Z}$

Examples:

(a)Integer Quantum Hall Insulator, (b) Quantum Spin Hall Insulator,

(c) 3d Z_2 Topological Insulator, (d) Spinless chiral p-wave (p+ip) superconductor (Moore

(e)Chiral d-wave $(d_{x^2-y^2} + i d_{xy})$ superconducto $(p_x + ip_y)_\uparrow \otimes (p_x - ip_y)_\downarrow$ superc

(g) ^{3}He B phase.

Reordered Table

Kitaev, arXiv:0901.2686

	TRS	PHS	chS		d=1	d=2	d=3	
A	0	0	0	unitary	0	Z	0	complex
AIII	0	0	1	chiral unitary	Z	0	Z	K-theory
AI	+1	0	0	orthogonal	0	0	0	
BDI	+1	+1	1	chiral orthogonal	Z	0	0	
D	0	+1	0	(triplet) SC	Z_2	Z	0	
DIII	−1	+1	1	triplet SC	Z_2	Z_2	Z	real
AII	−1	0	0	symplectic	0	Z_2	Z_2	K-theory
CII	−1	−1	1	chiral symplectic	Z	0	Z_2	
C	0	−1	0	singlet SC	0	Z	0	
CI	+1	−1	1	singlet SC	0	0	Z	

Periodic table for topological insulators

Classification in any dimension

Bott periodicity:

$$\tilde{K}_{\mathbb{C}}^{n+2}(X) \cong \tilde{K}_{\mathbb{C}}^{n}(X)$$

$$\tilde{K}_{\mathbb{R}}^{n+8}(X) \cong \tilde{K}_{\mathbb{R}}^{n}(X)$$

Summary

- Many topological insulators of non-interacting fermions have been found.

  interacting fermions??

- Gapless boundary modes (Dirac or Majorana)

  stable against any (weak) perturbation disorder

- Majorana fermions

  to be found experimentally in solid-state devices