Time-dependent backgrounds of compactified 2D string theory: complex curve and instantons

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Plan

- Non-perturbative effects (NPE) and in minimal string theories, $c = 1 - \frac{6}{p(p+1)}$
  - Branes in minimal string theories: FZZT and ZZ
  - String backgrounds as complex curves: ZZ brans and double points, compact and non-compact $A$ and $B$ cycles
  - Comparison with the NPE in Matrix Models: ambiguity problem

- Non-perturbative effects in $c = 1$ string theory
  - CFT - MM dictionary
  - Matrix Quantum Mechanics: Chiral quantization. Complex curve and NPE.
  - Tachyon perturbations as deformations of the complex curve.
  - Leading and subleading non-perturbative corrections. Equation for the double points.
Sketch of the non-perturbative effects in string theory:

String partition function:

\[ \mathcal{F} \simeq \mathcal{F}_{\text{pert}} + \mathcal{F}_{\text{nonpert}} \]
\[ = \sum_{h \geq 0} F_h g_s^{2h-2} + \sum_{n \geq 0} A_n e^{-D_n/g_s} \]

\[ \uparrow \quad \text{closed strings} \quad \uparrow \quad \text{open strings} \]

\[ \mathcal{F}_h = \text{vacuum } h\text{-loop closed string amplitudes} \]
\[ D_n = \text{disk amplitudes with Dirichlet b.c. along ‘D-branes’} \]

The contribution of a D-brane =

\[ 1 + \frac{1}{g_s} + \frac{1}{2} \frac{1}{g_s^2} + \frac{1}{2} + \ldots = \exp \left( \frac{1}{g_s} + \frac{1}{2} + \ldots \right) \]
Branes in minimal \((c < 1)\) string theories:

Minimal string theory = \text{Liouville} + (p, q) \text{Minimal “matter” CFT} + \text{Ghosts}

In Cardy’s formalism: brane = local boundary condition (Liouville + Matter)

- `Matter’ branes: \((r; s)\) boundary conditions \(1 \leq r \leq p - 1; 1 \leq s \leq q - 1\)
  
  [Cardy ’89]

- `Liouville’ branes:
  
  - Extended branes or FZZT branes [Fateev, Zamolodchikov\(^2\), Teschner]
    
    – Labeled by the “boundary cosmological constant” \(\mu_B \in \mathbb{R}\)

  - D-branes localized ‘at infinity’ or \((m, n)\) ZZ branes
    
    – Labeled by \(m, n \in \mathbb{N}\)
    
    [A\& Al. Zamolodchikov’00]

- D-instanton = \((m, n)\) Liouville brane \(\times (r, s)\) minimal model brane
  
  [Martinec ’03, Klebanov-Martinec-Seiberg’03]
FZZT, ZZ and algebraic curve

Disk partition function on FZZT brane: \( \Phi = \Phi(\mu_B) \)

\[
x \equiv \mu_B = \sqrt{\mu} \cosh \tau, \quad y \equiv \frac{d\Phi}{d\mu_B} = \mu^{q/2p} \cosh \left( \frac{q}{p} \tau \right),
\]

Geometrical interpretation of the ZZ-branes \[\text{[Seiberg&Shih'03]}\]:

\( x \) and \( y \) define an algebraic curve \( F(x, y) = 0 \), globally parametrized by the boundary parameter \( \tau \)

ZZ-branes \( \leftrightarrow \) double points of the curve: \( x_{m,n} = x_{m,-n}, \ y_{m,n} = y_{m,-n} \).

Based on a (still poorly understood) linear relation between FZZT and ZZ states

\[
\Phi_{mn}^{ZZ} = \Phi^{FZZT}(\tau_{m,n}) - \Phi^{FZZT}(\tau_{m,-n}), \ \tau_{m,n} = im + \frac{q}{p}n
\]
Moduli of the complex curve

Example: The theory $(4, 5)$.

- singular points at $x = \infty$ and $y = \infty$
- related by a ‘non-compact B-cycles’

Moduli associated with the $A$ and $B$ cycles:

\[ \oint_A ydx = \mu \quad \text{– cosmological constant} \]
\[ \oint_B ydx = \partial_\mu \mathcal{F} \quad \text{– the 1-point function of the ‘puncture operator’} \]
\[ \oint_{B_{mn}} ydx = \Gamma_{mn} \quad \text{– the instanton ‘chemical potential’} \]
\[ \int_x^\infty ydx = \Phi(x) \quad \text{– the disc partition function with FZZT b.c.} \]
Example: The theory \((4, 5)\).

Double points \(x_{mn}, y_{mn}\) connected by compact \(B_{mn}\) cycles.

Moduli associated with the \(A\) and \(B\) cycles:

\[
\int_A ydx = \mu \quad - \text{cosmological constant}
\]

\[
\int_B ydx = \partial_\mu F = \text{the 1-point function of the 'puncture operator'}
\]

\[
\int_{B_{mn}} ydx = \Gamma_{mn} \quad - \text{the instanton 'chemical potential'}
\]

\[
\int_x^\infty ydx = \Phi(x) \quad - \text{the disc partition function with FZZT b.c.}
\]
Degeneracy problem at the \((p, q)\) critical point:

\[
\mathcal{D}_{mn} = \mathcal{Z}^{ZZ}_{(m,n),(11)} = \mathcal{Z}^{ZZ}_{(1,1)(m,n)} = \mathcal{Z}^{ZZ}_{(m,1),(1n)} = \mathcal{Z}^{ZZ}_{(1,n),(m,1)}
\]

- the degeneracy is not lifted by perturbations \([\text{V. Kazakov & I.K.'04}]\)
- possible solution: all 4 states are physically identical \([\text{Seiberg& Shih}]\)

The degeneracy might be understood by studying finite perturbations off the \((p, q)\) critical points.

Then the string theory does not factorize to Matter \(\times\) Liouville. No-world sheet description known for such string theories.

The \(c = 1\) string theories are more interesting: \(\exists\) solvable models that do not reduce to Matter \(\times\) Liouville: sine-Liouville, ‘cigar’, ‘paperclip’ etc., with interesting applications in string theory.

\(\Rightarrow\) It is potentially important to study the non-perturbative effects in the \(c = 1\) matrix models in non-trivial (time-dependent) backgrounds.
$c = 1$ string theory: World sheet CFT

Compactified Euclidean $c = 1$ string theory:

$\chi$ – matter field $\rightarrow$ Euclidean compactified time: $\chi + 2\pi R \equiv \chi$

$\phi$ – Liouville field $\rightarrow$ space

The action for the linear dilaton background:

$$S_{c=1} = \frac{1}{4\pi} \int d^2\sigma \left[ (\partial \chi)^2 + (\partial \phi)^2 + 2\hat{R}\phi + \mu \phi e^{2\phi} + \text{ghosts} \right]$$

‘Tachyon’ operators:

$$\mathcal{T}_q \sim \int d^2\sigma \ e^{iq\chi} e^{(2-|q|)\phi}; \quad q_k = k/R, \ k = \pm 1, \pm 2, \ldots.$$ 

Perturbations by tachyon modes: $\delta S = \sum_k (t_k \mathcal{T}_{k/R} + t_{-k} \mathcal{T}_{-k/R})$

E.g. sine-Liouville perturbation: $\delta S_{SL} = \lambda \int d^2\sigma \ \cos(\chi/R) \ e^{(2-\frac{1}{R})\phi}, \ \lambda = t_1 = t_{-1}$
\( c = 1 \) string theory: Matrix model (Matrix Quantum Mechanics)

- Singlet sector of MQM = free fermions in upside-down gaussian potential \( U(x) = -\frac{1}{2}x^2 \).

- Hamiltonian: \( H = \frac{1}{2}(p^2 - x^2) \)

\[
\begin{align*}
\mathcal{T}_q &\leftrightarrow e^{-qt} \text{ Tr } x^{|q|}, \quad (q > 0) \\
\mathcal{T}_q &\leftrightarrow e^{-qt} \text{ Tr } x^{|q|}, \quad (q < 0)
\end{align*}
\]

\[
x_\pm = \frac{1}{\sqrt{2}}(x \pm p)
\]

- The ground state (fermi sea filled up to \( E = -\mu \)) describes the linear dilaton background.

- The profile of the fermi sea \( p^2 - x^2 = -\mu \) \( \rightarrow \) equation of a complex curve

- Global parametrization: \( x = \sqrt{2\mu} \cosh \tau, \quad p = \sqrt{2\mu} \sinh \tau \) \( (\tau \in \mathbb{C}) \)

- **SL deformation of MQM** [G.Moore’92]; **Integrability and Toda flows** [Jevicki, Moore-Plesser] [V. Alexandrov, V. Kazakov, I.K., D. Kutasov]

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Time-dependent backgrounds of compactified 2D string theory: – p.9/25
Chiral quantization of MQM

- Light-cone coordinates in the phase space: \( x_{\pm} = \frac{1}{\sqrt{2}} (x \pm p) \)
  
  \( x_{\pm} \) can be identified (in some sense) with the generators of the ground ring,
  
  \( x_+ x_- = \mu \) – the relation of the ground ring [Witten'91]

- 1-particle wavefunctions in “\( x_+ \)” and “\( x_- \)” representations:

  \[
  H \psi^E_\pm = E \psi^E_\pm; \quad H = -i(x_+ \partial x_+ + 1/2) = i(x_- \partial x_- + 1/2)
  \]

  \[
  \Rightarrow \quad \psi^E_+(x_+) = \frac{1}{\sqrt{\pi}} e^{-i\phi_0/2} x_+^{E-\frac{1}{2}}, \quad \psi^E_-(x_-) = \frac{1}{\sqrt{\pi}} e^{i\phi_0/2} x_-^{-E-\frac{1}{2}}
  \]

- The potential \( U(x) = -x^2/2 \) is replaced by branch point at \( x_{\pm} = 0 \)

- Scattering matrix \( \Leftrightarrow \) Fourier operator \( S \).
  
  The scattering phase \( \phi_0 = \phi_0(E) \) is determined by the condition \( S = I \): 

  \[
  \psi^E_-(x_-) = [S \psi^E_+](x_-) \equiv \frac{1}{\sqrt{2\pi}} \int dx_+ e^{i x_+ + x_-} \psi^E_+(x_+)
  \]
scattering phase ⇒ density of states ⇒ partition function

- **Scattering phase** \( \phi_0(E) \):

\[
\begin{align*}
\mathbb{S} \psi_+^E &= \psi_-^E & \Rightarrow & \quad \langle E|\mathbb{S}|E \rangle = \rho_0(E) \approx \frac{1}{2\pi} \log \Lambda \\
\Rightarrow & \quad e^{i\phi_0(E)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\pi}{2}(E-i/2)} \Gamma(iE + 1/2).
\end{align*}
\]

- Small imaginary part \( \text{Im} \phi_0(-\mu) = \frac{1}{2} \log(1 + e^{-2\pi\mu}) \)

- **Density of states**:

\[
\rho(E) = \frac{\log \Lambda}{2\pi} - \frac{1}{2\pi} \frac{d\phi_0(E)}{dE}
\]

- **Grand canonical partition function**:

\[
\mathcal{F}(\mu, \beta) = \int_{-\infty}^{\infty} dE \rho(E) \log \left(1 + e^{-\beta (\mu + E)}\right), \quad \beta = 2\pi R
\]

\[
\Rightarrow 2 \sin \frac{\partial \mu}{2R} \cdot \mathcal{F}(\mu) = \phi_0(-\mu)
\]
Non-perturbative corrections to the free energy

In the integral representation of $\phi_0$

$$\phi_0(-\mu) = -\frac{i}{2} \int_{\frac{1}{\Lambda}}^{\infty} ds \frac{e^{i\mu s}}{s \sinh \frac{s}{2}}$$

the non-perturbative part comes from the poles: $\phi_0 = \phi_0^{\text{pert}} + \phi_0^{\text{np}}$

$$\phi_0^{\text{pert}} = -\mu \log \mu + \mu + \mathcal{O}(1/\mu^2), \quad \phi_0^{\text{np}} = -i \sum_n \frac{1}{2n} (-1)^n e^{-2\pi n \mu}.$$

The non-perturbative piece of the free energy:

$$\mathcal{F}_n^{\text{np}}(\mu)\{t_k=0\} = i \sum_n \frac{e^{-2\pi n \mu}}{4n(-1)^n \sin \frac{\pi n}{R}} + i \sum_n \frac{e^{-2\pi R n \mu}}{4n(-1)^n \sin(\pi R n)}.$$

– no pre-factors $g_{s}^{1/2} \sim \mu^{-1/2}$, unlike the minimal string theories! Why?
Flat direction $\rightarrow$ No isolated saddle points

Evaluate the integral quasiclassically:

$$
\langle E|S|E \rangle = \int 0 \sqrt{\Lambda} \frac{dx}{x-x} e^{ix} + x + iE \log x + x - i\phi_0(E) = \sqrt{2\pi} \log \Lambda
$$

No saddle points but "saddle contours":

$$
x_+ x_- = -E e^{2\pi in}, \quad n = 0, 1, 2, ...
$$

Dominant saddle ($n = 0$) $\leftrightarrow$ Complex curve:

$$
x_+ x_- = \mu \quad \leftarrow \quad x_\pm(\tau) = \sqrt{\mu} e^{\pm \tau}, \quad \tau \in \mathbb{C}
$$

Subdominant saddles ($n = 1, 2, ...$) $\leftrightarrow$ Instantons

$\leftrightarrow$ ‘double contours’ $\{x_\pm(\tau'), x_\pm(\tau'') | \tau' \in \mathbb{R} + \pi in, \tau'' \in \mathbb{R} - \pi in\}$
Tachyon perturbations can be introduced by changing the asymptotics of the fermionic wave functions:

\[ \Psi_{\pm}^{E}(x_{\pm}) = x_{\pm}^{{\pm i E} - \frac{1}{2}} \exp \left[ \mp i \left( \frac{1}{2} \phi(E) + \sum_{k=1}^{k_{\text{max}}} t_{\pm k} x_{\pm}^{k/R} + \ldots \right) \right] \quad x_{\pm} \to \infty \]

The saddle point equations for the integral \( \langle E|\mathcal{S}|E \rangle = \frac{1}{2\pi} \log \Lambda \) gives asymptotic conditions for the complex curve at \( x_{\pm} \to \infty \):

\[
\begin{align*}
x_{+}x_{-} &= \mu + \frac{1}{R} \sum_{k \geq 1} k t_{k} x_{+}^{k/R} + \ldots \quad (x_{+} \to \infty) \\
&= \mu + \frac{1}{R} \sum_{k \geq 1} k t_{-k} x_{-}^{k/R} + \ldots \quad (x_{-} \to \infty)
\end{align*}
\]

E.g. for Sine-Liouville perturbation, \( t_{1} = t_{-1} = \lambda \), \( \partial_{\mu}^{2} \mathcal{F} = -2R \log u \):

\[
x_{\pm} = u e^{\pm \tau} + \frac{\lambda}{R} u^{\frac{1}{2R}} e^{\pm \tau(1-1/R)}, \quad u^{2} + \frac{R-1}{R^{3}} \lambda^{2} u^{\frac{2}{R}-2} = \mu
\]
Sub-dominant saddles = double points of the deformed curve

For the deformed curve the degeneracy is lifted:

Double points: \( x_+ (\tau') = x_+ (\tau'') \) and \( x_- (\tau') = x_- (\tau'') \)

\[
\tau' = -i\theta_n, \quad \tau'' = i\theta_n, \quad \sin (\theta_n) = \frac{\lambda}{R} u \frac{1}{2\pi} \sin \left( \frac{1-R}{R} \theta_n \right),
\]

Quasiclassical wave functions:

\[
\Psi^E_\pm (x_\pm) \approx \frac{1}{\sqrt{\pm 2\pi \partial_\tau x_\pm}} e^{\mp i\phi/2} \exp \left( \mp i \int_\infty^{x_\pm} x_\mp dx_\mp \right)
\]

\[
\langle E|\mathcal{S}|E \rangle = \frac{1}{2\pi} \log \Lambda \quad \Rightarrow \quad e^{i\phi_{\text{pert}}} \left[ \log \Lambda + \sum_{n=1}^{\infty} \sqrt{\frac{\pi}{2}} \Delta_n e^{-\Gamma_n} \right] = e^{i\phi(-\mu)} \log \Lambda
\]

\[
\phi_{\text{pert}} = \int_{-\infty}^{\infty} x_- (\tau) \, dx_+ (\tau), \quad \Gamma_n = i \int_{-i\theta}^{i\theta} x_- (\tau) \, dx_+ (\tau)
\]

\[
\Delta_n = \left| \left( \frac{\partial x_+}{\partial \tau} \right)_{-i\theta_n} \left( \frac{\partial x_-}{\partial \tau} \right)_{i\theta_n} \left( 1 - \left( \frac{dx_-}{dx_+} \right)_{-i\theta_n} \left( \frac{dx_+}{dx_-} \right)_{i\theta_n} \right) \right|^{-1/2}
\]
Subleading NPE for the free energy

\[ \mathcal{F} \approx \mathcal{F}_{\text{pert}} + \frac{i\sqrt{\pi}}{2\sqrt{2}\log \Lambda} \sum_{n>0} \frac{\Delta_n}{\sin \frac{\partial \mu S_n}{2R}} e^{-S_n}. \]

\[ A_n \sim \frac{\Delta_n}{\sin \frac{\partial \mu S_n}{2R}} = \left( \sin^2 \frac{\theta}{R} \left[ \left( \frac{\partial x_+}{\partial \tau} \right)_{-i\theta_n} \left( \frac{\partial x_-}{\partial \tau} \right)_{i\theta_n} + \left( \frac{\partial x_+}{\partial \tau} \right)_{i\theta_n} \left( \frac{\partial x_-}{\partial \tau} \right)_{-i\theta_n} \right] \right)^{-1/2}. \]

For sine-Liouville \( t_1 = t_{-1} = \lambda \):

\[ A(\xi, y) = \frac{C}{\sqrt{e^{-\frac{1}{R^2}} \sin^2 \theta \sin^2 \frac{\theta}{R} \left( (\frac{1}{R} - 1) \cot \left( \frac{1-R}{R} \theta \right) - \cot \theta \right)}} \]

\[ C = i \frac{\sqrt{\pi R}}{4\sqrt{2}\log \Lambda} \]

The pre-exponents scale as \( g_s^{1/2} \) and are non-analytic in the limit \( \lambda \to 0 \).
Bosonization in the perturbed theory

Tachyon modes = collective excitations of fermions (left and right moving waves on the Fermi surface). After second quantization these modes form a bosonic field

In \((x_+, x_-)\) representation the fermions can be exactly bosonized.

Scnd quantized fermion fields:
\[
\hat{\Psi}_\pm(x_\pm, t) = \int dE \ e^{\mp Et/2} \ \Psi^E_\pm \left( e^{\mp t} x_\pm \right) \ b(E),
\]
\[
\hat{\Psi}_\pm\dagger(x_\pm, t) = \int dE \ e^{\mp Et/2} \ \Psi^E_\pm \left( e^{\mp t} x_\pm \right) b\dagger(E), \quad \{b\dagger(E), b(E')\} = \delta(E - E')
\]

Thermal vacuum:
\[
\langle \mu | b\dagger(E) b(E') | \mu \rangle = \frac{\delta(E - E')}{1 + e^{\beta(\mu + E)}}.
\]

Fermion phase space density operator:
\[
\hat{U}(x_+, x_-, t) = \frac{1}{\sqrt{2\pi}} \ e^{ix_+ + x_-} \ \hat{\Psi}_-\dagger(x_-, t) \hat{\Psi}_+(x_+, t)
\]
Bosonization rules

**Phase → Bosonic field:**
\[ \Phi_{\pm}(x_{\pm}) = V_{\pm}(x_{\pm}) + \frac{1}{2} \phi - E \log x_{\pm} + D_{\pm}(x_{\pm}), \]

\[ V_{\pm}(x_{\pm}) = \sum_{n>0} t_{\pm n} x_{\pm}^{n/R}, \quad \phi = -\frac{1}{R} \partial E \quad D_{\pm}(x_{\pm}) = \sum_{n \geq 1} \frac{1}{n} x_{\pm}^{-n/R} \partial t_{\pm n}. \]

**Fermion wave function → Bekker-Akhieser function**
\[ \Psi_{\pm}^{-\mu}(x_{\pm}) = (2\pi x_{\pm})^{-1/2} \left< \mu \right| e^{-i\Phi_{\pm}(x_{\pm})} \left| \mu \right> \]

**Normal product sign (\(\dot{\cdot}\)) – all derivatives moved to the right**
**Expectation value:**
\[ \langle \hat{O} \rangle = \mathcal{Z}^{-1} \cdot \hat{O} \cdot \mathcal{Z}. \]

**Operator formula for the fermion bilinears:**
\[ \Psi_{\pm}^{\dagger}(x_{-}) \Psi_{+}(x_{+}) = \frac{1}{2\pi R} \left( x_{+} x_{-} \right) - \frac{R+1}{2R} e^{-i\Phi(x_{+},x_{-})}, \]

where \[ \Phi(x_{+},x_{-}) = \Phi_{+}(x_{+}) + \Phi_{-}(x_{-}). \]
Pre-exponential factor as ‘annulus amplitude’

\[
\left\langle : \hat{\Phi}(x_+, x_-) \hat{\Phi}(y_+, y_-) : \right\rangle_c = \log \sinh \frac{\tau(y_-) - \tau(y_+)}{2R} \sinh \frac{\tau(x_-) - \tau(x_+)}{2R} \\
\quad + \log(x_-^{1/R} - y_-^{1/R}) + \log(x_+^{1/R} - y_+^{1/R}),
\]

\[
\left\langle \mu | \hat{\Psi}^\dagger(x_-) \hat{\Psi}(x_+) | \mu \right\rangle =
\]

\[
= \frac{1}{2\pi R} \frac{1}{(x_+ x_-) \frac{R+1}{2R}} \exp \left[ -i\Phi(x_+, x_-) - \frac{1}{2} \left\langle : \hat{\Phi}(x_+, x_-) \hat{\Phi}(x_+, x_-) : \right\rangle_c \right]
\]

\[
= e^{-i\Phi(x_+, x_-)} \frac{1}{2\pi \sqrt{\partial_\tau x_+ \partial_\tau x_-}} \frac{1}{2R \sinh \frac{\tau(x_-) - \tau(x_+)}{2R}}.
\]
NPC from bosonization

The denominator diverges on the surface of the Fermi sea, \( \tau(x_+) = \tau(x_-) \), where the quasiclassics breaks down

\[
2R \sin\left(\frac{1}{2R} \partial_\mu\right) \left\langle \mu | \hat{\Psi}_-^{\dagger}(x_-) \hat{\Psi}_+(x_+) |\mu \right\rangle = \frac{e^{-i\Phi(x_+,x_-)}}{2\pi \sqrt{-\partial_\tau x_+ \partial_\tau x_-}} = \frac{\Psi_-^{\mu}(x_-) \Psi_+^{\mu}(x_+)}{2\pi \sqrt{-\partial_\tau x_+ \partial_\tau x_-}}
\]

\[
N \equiv \langle \mu | \text{Tr} \mathbb{P} |\mu \rangle = \int dx_+ dx_- \langle \mu | \hat{U}(x_+,x_-) |\mu \rangle = -\frac{\mu}{2\pi} \log \Lambda - \frac{1}{2\pi R} \partial_\mu \mathcal{F},
\]

\[
\sin\left(\frac{1}{2R} \partial_\mu\right) N = -\frac{1}{4\pi R} (\log \Lambda + \partial_\mu \phi(-\mu)).
\]

\[
\Rightarrow \frac{1}{\sqrt{2\pi}} \int dx_+ dx_- \frac{e^{ix_+ x_- - i\Phi(x_+,x_-)}}{\sqrt{-\partial_\tau x_+ \partial_\tau x_-}} = \log \Lambda.
\]

The evaluation of this integral by the saddle point method gives the non-perturbative corrections to the zero mode \( \phi \) of the bosonic field.
Several fermion bilinears

\[ \langle \mu | \prod_{j=1}^{n} \Psi_-(x_j^-) \Psi_+(x_j^+) | \mu \rangle = \]

\[ = \prod_{k=1}^{n} \frac{e^{-i \Phi(x^+_k, x^-_k)}}{2\pi \sqrt{\partial_\tau x^+_k \partial_\tau x^-_k}} \text{det}_{i,j} \left( \frac{1}{2R \sinh \left( \frac{\tau(x^+_i) - \tau(x^-_j)}{2R} \right)} \right). \]

exact (?) after \( \omega_\pm = e^\tau \rightarrow e^{i\partial_\mu} \)

- The parameter \( \tau \) plays the same role as the boundary parameter parametrizing the boundary cosmological constant \( \mu_B = \sqrt{\mu} \cosh \tau \) in Liouville theory.

- Here there are two “boundary cosmological constants”: \( x_+(\tau) \) and \( x_-(\tau) \).

- Two “disk amplitudes" \( \Phi_+(x_+) = \int_{x_+}^{x} dx_+ \) and \( \Phi_-(x_-) = \int_{x_-}^{x} dx_- \) related by Legendre transformation
Back to \((x, p)\) representation

1) Stationary background

\[ x_+(\tau), x_-(\tau) \rightarrow x(\tau), w(\tau) = d\Phi/dx \]

\(\Phi(x)\) - the disk partition function with Dirichlet matter \(\times\) FZZT \(\mu_B = x\).

\[
\begin{align*}
   x_+(\tau) + x_-(\tau) &= 2x(\tau), \\
   x_+(\tau) - x_-(\tau) &= w(\tau + i\pi) - w(\tau - i\pi)
\end{align*}
\]

The curve

\[
x(\tau) = \sqrt{2\mu} \cosh \tau, \quad w(t) = -\frac{\sqrt{2\mu}}{\pi} \tau \sinh \tau
\]

is a \(\mathbb{Z}_2\) orbifold of the universal cover of the hyperboloid \(x_+x_- = \mu\), obtained by identifying the points \(\tau\) and \(-\tau\).

The branch points of \(y = w(x)\) are images of infinitely many points \(\tau = \pm i\pi n\), the (degenerate) limit of the double points of the minimal string curves

\[
\tau_{mn} = i(m/b + nb), \quad \tau_{m,-n} = i(m/b - nb)
\]
Back to \((x, p)\) representation

2) Time-dependent background

\[
x(\tau) = \sqrt{2} e^{-\frac{\tau}{2R}} \left[ \cosh \tau + \sum_{k=1}^{k_{\text{max}}} a_k \cosh \left( (1 - \frac{k}{R}) \tau \right) \right],
\]
\[
p(\tau) = \sqrt{2} e^{-\frac{\tau}{2R}} \left[ \sinh \tau + \sum_{k=1}^{k_{\text{max}}} a_k \sinh \left( (1 - \frac{k}{R}) \tau \right) \right].
\]

\[
p(\tau) = \frac{w(\tau + i\pi_1(\tau)) - w(\tau - i\pi_1(\tau))}{2i},
\]

where the function \(\pi_1(t)\) is obtained as the solution of the transcendental equation

\[
\sin \pi_1 + \sum_{k=1}^{k_{\text{max}}} a_k \frac{\sinh \left( (1 - \frac{k}{R}) t \right)}{\sinh t} \sin \left( (1 - \frac{k}{R}) \pi_1 \right) = 0, \quad t \in \mathbb{R}.
\]

As a consequence, the parameter \(\tau\) does not uniformize the curve \(y = w(x)\). In general, \(w(\tau)\) will have infinitely many branch points.
horizontal: $\text{Im} \tau$
vertical: $t = \text{Re} \tau$

The lines give
the solutions of the
transcendental eqn
for $\pi_k(t)$
Conclusion

- The time-dependent backgrounds of the $c = 1$ string theory are described by complex curves with two singular points.

- All NPC are associated with “double points” (=vanishing cycles) of the complex curve.

- The leading exponents and the subleading factors of the non-perturbative corrections are expressed in terms of the one- and two-point functions of a bosonic field defined on the complex curve.

- The subleading correction is non-analytic in the perturbation couplings.

Questions:

- Is there a D-brane interpretation of the correlation functions of the fermion bilinears?

- If so, identify the CFT boundary conditions for these ‘chiral branes’. For $\phi$ large and negative, these branes are extended in $\chi + \phi$ direction and localized in $\chi - \phi$ direction, or vice versa.

\[
\langle \Phi(x_+, x_-) \rangle = \left\langle \sum_{n \geq 1} \left[ \mathcal{T}_{n/R} x_+^{n/R} + \mathcal{T}_{-n/R} x_-^{n/R} \right] \right\rangle_{\text{sphere}} = \langle ??? \rangle_{\text{disk}}
\]