Time-dependent backgrounds of compactified 2D string theory:

complex curve and instantons

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Plan

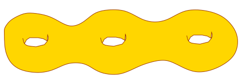
- Non-perturbative effects (NPE) and in **minimal string theories,** $c = 1 - \frac{6}{p(p+1)}$
  - Branes in minimal string theories: FZZT and ZZ
  - String backgrounds as complex curves: ZZ brans and double points, compact and non-compact A and B cycles
  - Comparison with the NPE in Matrix Models: ambiguity problem
- Non-perturbative effects in $c = 1$ **string theory**
  - CFT - MM dictionary
  - Matrix Quantum Mechanics: Chiral quantization. Complex curve and NPE.
  - Tachyon perturbations as deformations of the complex curve.
  - Leading and subleading non-perturbative corrections. Equation for the double points.
  - Exact bosonization of MQM in chiral quantization. Bosonization rules. Vertex operators and D-instantones.

Sketch of the non-perturbative effects in string theory:

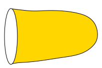
String partition function:

$$\begin{aligned}\mathcal{F} \;&\simeq\; \mathcal{F}_{\rm pert} \;+\; \mathcal{F}_{\rm nonpert}\\ &= \sum_{h\geq 0} F_h\, g_s^{2h-2} + \sum_{n\geq 0} A_n\, e^{-\mathcal{D}_n/g_s}\end{aligned}$$

↑ closed strings ↑ open strings



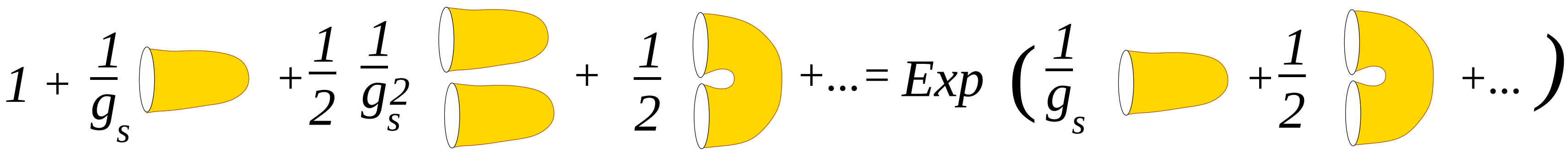
$\mathcal{F}_h$ = vacuum h-loop closed string amplitudes



$\mathcal{D}_n$ = disk amplitudes with Dirichlet b.c. along 'D-branes'

[Polchinski '94]

The contribution of a D-brane =



Branes in minimal ($c < 1$) string theories:

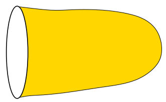
Minimal string theory = Liouville + (p, q) Minimal "matter" CFT + Ghosts

In Cardy's formalism: brane = local boundary condition (Liouville $+$ Matter)

- 'Matter' branes: (r; s) boundary conditions $1 \leq r \leq p-1; 1 \leq s \leq q-1$
  [Cardy '89]
- 'Liouville' branes:
  - Extended branes or FZZT branes [Fateev, Zamolodchikov^2, Teschner]
    – Labeled by the "boundary cosmological constant" $\mu_B \in \mathbb{R}$
  - D-branes localized 'at infinity' or (m, n) ZZ branes
    [A& Al. Zamolodchikov'00]
    – Labeled by $m, n \in \mathbb{N}$
- D-instanton = (m, n) Liouville brane $\times$ (r, s) minimal model brane
  [Martinec '03,Klebanov-Martinec-Seiberg'03]

FZZT, ZZ and algebraic curve

- Disk partition function on FZZT brane: $\Phi = \Phi(\mu_B)$



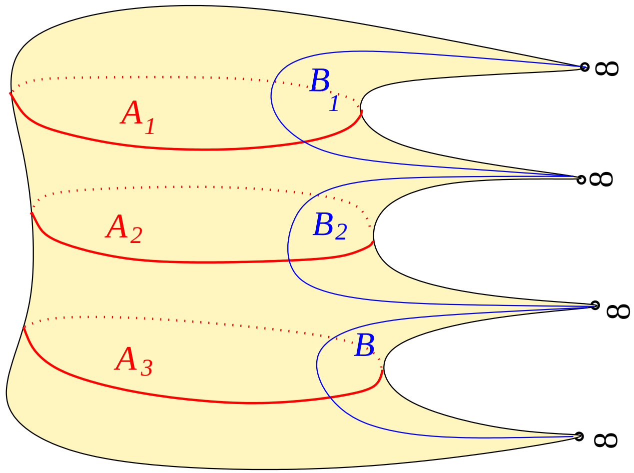
$$x \equiv \mu_B = \sqrt{\mu}\cosh\tau, \qquad y \equiv \frac{d\Phi}{d\mu_B} = \mu^{q/2p}\cosh\left(\frac{q}{p}\tau\right),$$

- Geometrical interpretation of the ZZ-branes [Seiberg&Shih'03]:
  - x and y define an algebraic curve $F(x,y)=0$, globally parametrized by the boundary parameter τ
  - ZZ-branes $\leftrightarrow$ double points of the curve: $x_{m,n} = x_{m,-n}, \;\; y_{m,n} = y_{m,-n}$.
- Based on a (still poorly understood) linear relation between FZZT and ZZ states

$$\Phi^{ZZ}_{mn} = \Phi^{FZZT}(\tau_{m,n}) - \Phi^{FZZT}(\tau_{m,-n}), \quad \tau_{m,n} = im + i\frac{q}{p}n$$

Moduli of the complex curve

Example: The theory $(4,5)$.



singular points at $x = \infty$ and $y = \infty$ related by a 'non-compact B-cycles' $\longleftarrow$

. .

Moduli associated with the A and B cycles :

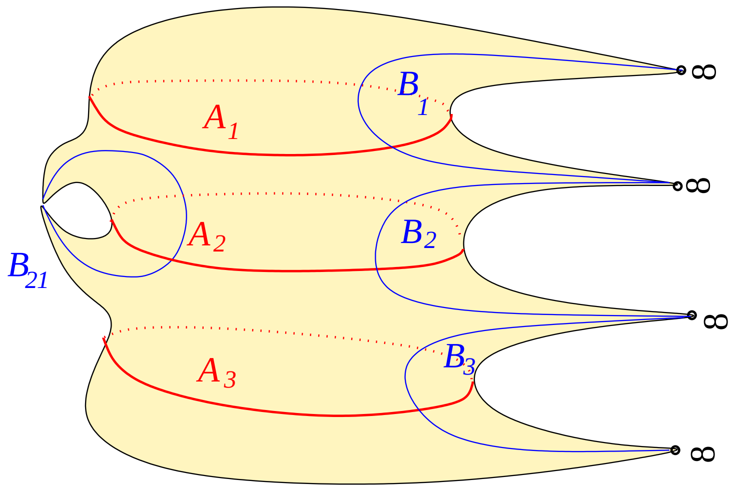
$\oint_A ydx = \mu$ – cosmological constant
$\int_B ydx = \partial_\mu \mathcal{F}$ = the 1-point function of the 'puncture operator'
$\int_{B_{mn}} ydx = \Gamma_{mn}$ – the instanton 'chemical potential'

$\int_x^\infty ydx = \Phi(x)$ – the disc partition function with FZZT b.c.

Moduli of the complex curve

Example: The theory $(4, 5)$.

double points x_{mn}, y_{mn} connected by compact B_{mn} cycles ⟶



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Algebraic curve beyond FZZ/ZZ

- Degeneracy problem at the (p,q) critical point:
  $$\mathcal{D}_{mn} = \mathcal{Z}^{ZZ}_{(m,n),(11)} = \mathcal{Z}^{ZZ}_{(1,1)(m,n),} = \mathcal{Z}^{ZZ}_{(m,1),(1n)} = \mathcal{Z}^{ZZ}_{(1,n),(m,1)}$$
  – the degeneracy is not lifted by perturbations [V. Kazakov & I.K.'04]
  – possible solution: all 4 states are physically identical [Seiberg& Shih]
- The degeneracy might by understood by studying finite perturbations off the (p,q) critical points.

  Then the string theory does not factorize to Matter $\times$ Liouville. No-world sheet description known for such string theories.
- The $c=1$ string theories are more interesting: $\exists$ solvable models that do not reduce to Matter $\times$ Liouville: sine-Liouville, 'cigar', 'paperclip' etc., with interesting applications in string theory.

  $\Rightarrow$ It is potentially important to study the non-perturbative effects in the $c=1$ matrix models in non-trivial (time-dependent) backgrounds.

$c = 1$ string theory: World sheet CFT

Compactified Euclidean $c = 1$ string theory:

χ – matter field $\quad \to$ Euclidean compactified time: $\quad \chi + 2\pi R \equiv \chi$
ϕ – Liouville field $\quad \to$ space

The action for the linear dilaton background:

$$S_{c=1} = \frac{1}{4\pi} \int d^2\sigma \left[(\partial\chi)^2 + (\partial\phi)^2 + 2\hat{R}\phi + \mu\phi\, e^{2\phi} + \text{ghosts} \right]$$

'Tachyon' operators:

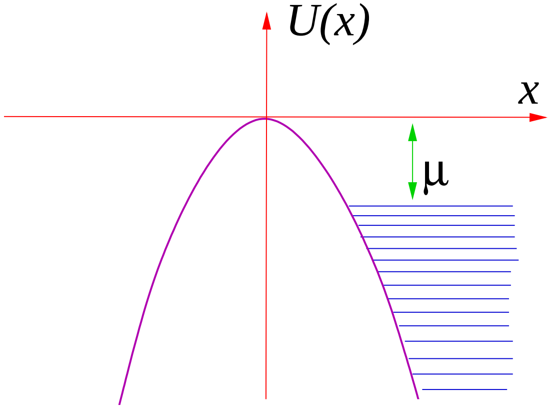
$$\mathcal{T}_q \sim \int d^2\sigma\, e^{iq\chi} e^{(2-|q|)\phi}; \qquad q_k = k/R, \;\; k = \pm 1, \pm 2, \dots .$$

Perturbations by tachyon modes: $\quad \delta S = \sum_k (t_k \mathcal{T}_{k/R} + t_{-k}\mathcal{T}_{-k/R})$

E.g. sine-Liouville perturbation: $\delta S_{SL} = \lambda \int d^2\sigma\, \cos(\chi/R)\, e^{(2-\frac{1}{R})\phi}, \;\; \lambda = t_1 = t_{-1}$

$c = 1$ string theory: Matrix model (Matrix Quantum Mechanics)

- Singlet sector of MQM = free fermions in upside-down gaussian potential $U(x) = -\frac{1}{2}x^2$.



$$\text{Hamiltonian}: \; H = \tfrac{1}{2}(p^2 - x^2)$$

$$\mathcal{T}_q \leftrightarrow e^{-qt} \, \mathrm{Tr}\, x_+^{|q|}, \quad (q > 0)$$

$$\mathcal{T}_q \leftrightarrow e^{-qt} \, \mathrm{Tr}\, x_-^{|q|}, \quad (q < 0)$$

$$x_\pm = \frac{1}{\sqrt{2}}(x \pm p)$$

- The ground state (fermi sea filled up to $E = -\mu$) describes the linear dilaton background.
- The profile of the fermi sea $p^2 - x^2 = -\mu \quad \to$ equation of a complex curve
- Global parametrization: $x = \sqrt{2\mu}\cosh\tau, \; p = \sqrt{2\mu}\sinh\tau \quad (\tau \in \mathbb{C})$
- SL deformation of MQM [G.Moore'92]; Integrability and Toda flows [Jevicki, Moore-Plesser] [V. Alexandrov, V. Kazakov, I.K., D. Kutasov]

Chiral quantization of MQM

- Light-cone coordinates in the phase space: $x_{\pm} = \frac{1}{\sqrt{2}}(x \pm p)$

  $x_{\pm}$ can be identified (in some sense) with the generators of the ground ring,

  $x_{+}x_{-} = \mu$ – the relation of the ground ring [Witten'91]

- 1-particle wavefunctions in "x_{+}" and "x_{-}" representations:

$$H\psi^{E}_{\pm} = E\psi^{E}_{\pm}; \quad H = -i(x_{+}\partial_{x_{+}} + 1/2) = i(x_{-}\partial_{x_{-}} + 1/2)$$

$$\Rightarrow \quad \psi^{E}_{+}(x_{+}) = \frac{1}{\sqrt{\pi}}\, e^{-i\phi_0/2}\, x_{+}^{iE-\frac{1}{2}}, \quad \psi^{E}_{-}(x_{-}) = \frac{1}{\sqrt{\pi}}\, e^{i\phi_0/2}\, x_{-}^{-iE-\frac{1}{2}}$$

- The potential $U(x) = -x^2/2$ is replaced by branch point at $x_{\pm} = 0$
- Scattering matrix $\Leftrightarrow$ Fourier operator $\mathbb{S}$.

  The scattering phase $\phi_0 = \phi_0(E)$ is determined by the condition $\mathbb{S} = \mathbb{I}$:

$$\psi^{E}_{-}(x_{-}) = [\mathbb{S}\psi^{E}_{+}](x_{-}) \equiv \frac{1}{\sqrt{2\pi}} \int dx_{+}\, e^{ix_{+}x_{-}}\, \psi^{E}_{+}(x_{+})$$

scattering phase ⇒ density of states ⇒ partition function

- Scattering phase $\phi_0(E)$:

$$\mathbb{S}\,\psi_+^E = \psi_-^E \quad \Rightarrow \quad \langle E|\mathbb{S}|E\rangle = \rho_0(E) \approx \tfrac{1}{2\pi}\log\Lambda$$

$$\Rightarrow \quad e^{i\phi_0(E)} = \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{\pi}{2}(E-i/2)}\,\Gamma(iE+1/2).$$

– Small imaginary part $\mathrm{Im}\,\phi_0(-\mu) = \frac{1}{2}\log(1+e^{-2\pi\mu})$

- Density of states:

$$\rho(E) = \frac{\log\Lambda}{2\pi} - \frac{1}{2\pi}\frac{d\phi_0(E)}{dE}$$

- Grand canonical partition function:

$$\mathcal{F}\,(\mu,\beta) = \int_{-\infty}^{\infty} dE\,\rho(E)\log\left(1+e^{-\beta(\mu+E)}\right), \qquad \beta = 2\pi R$$

$$\Rightarrow \quad \boxed{2\sin\tfrac{\partial_\mu}{2R}\cdot\mathcal{F}(\mu) = \phi_0(-\mu)}$$

Non-perturbative corrections to the free energy

In the integral representation of ϕ_0

$$\phi_0(-\mu) = -\frac{i}{2}\int_{\frac{1}{\Lambda}}^{\infty}\frac{ds}{s}\,\frac{e^{i\mu s}}{\sinh\frac{s}{2}}$$

the non-perturbative part comes from the poles: $\phi_0 = \phi_0^{\rm pert} + \phi_0^{\rm np}$

$$\phi_0^{\rm pert} = -\mu\log\mu + \mu + \mathcal{O}(1/\mu^2), \qquad \phi_0^{\rm np} = -i\sum_n \tfrac{1}{2n}\,(-1)^n\,e^{-2\pi n\mu}.$$

The non-perturbative piece of the free energy:

$$\mathcal{F}_{\rm np}(\mu)_{\{t_k=0\}} = i\sum_n \frac{e^{-2\pi n\mu}}{4n(-1)^n\sin\frac{\pi n}{R}} + i\sum_n \frac{e^{-2\pi R n\mu}}{4n(-1)^n\sin(\pi R n)}.$$

– no pre-factors $g_s^{1/2} \sim \mu^{-1/2}$, unlike the minimal string theories! Why?

Flat direction –> No isolated saddle points

Evaluate the integral quasiclassically:

$$\langle E|\mathbb{S}|E\rangle = \iint_0^{\sqrt{\Lambda}} \frac{dx_+ \, dx_-}{\sqrt{x_+ x_-}} \, e^{ix_+ x_- + iE\log x_+ x_- - i\phi_0(E)} = \sqrt{2\pi}\log\Lambda$$

No saddle points but "saddle contours":

$$x_+ x_- = -E\, e^{2\pi i n}, \quad n = 0, 1, 2, ...$$

Dominant saddle ($n = 0$) $\leftrightarrow$ Complex curve:

$$x_+ x_- = \mu \quad \Leftarrow \quad x_\pm(\tau) = \sqrt{\mu}\, e^{\pm\tau}, \quad \tau \in \mathbb{C}$$

subdominant saddles ($n = 1, 2, ...$) $\leftrightarrow$ Instantons

$\leftrightarrow$ 'double contours' $\{x_\pm(\tau'), x_\pm(\tau'') \mid \tau' \in \mathbb{R} + \pi i n, \ \tau'' \in \mathbb{R} - \pi i n\}$

Time-dependent backgrounds

Tachyon perturbations can be introduced by chanhing the asymptotics of the fermionic wave functions:

$$\Psi_{\pm}^{E}(x_{\pm}) = x_{\pm}^{\pm iE-\frac{1}{2}} \exp\left[\mp i\left(\frac{1}{2}\phi(E) + \sum_{k=1}^{k_{\max}} t_{\pm k}\, x_{\pm}^{k/R} + ...\right)\right] \qquad x_{\pm} \to \infty$$

The saddle point equations for the integral $\langle E|\mathbb{S}|E\rangle = \frac{1}{2\pi}\log\Lambda$ gives asymptotic conditions for the complex curve at $x_{\pm} \to \infty$:

$$\begin{aligned} x_{+}x_{-} &= \mu + \tfrac{1}{R}\sum_{k\geq 1} kt_{k}\, x_{+}^{k/R} + ... \qquad (x_{+} \to \infty) \\ &= \mu + \tfrac{1}{R}\sum_{k\geq 1} kt_{-k}\, x_{-}^{k/R} + ... \qquad (x_{-} \to \infty) \end{aligned}$$

E.g. fore Sine-Liouville perturbation, $t_1 = t_{-1} = \lambda$, $\partial_{\mu}^{2}\mathcal{F} = -2R\log u$:

$$x_{\pm} = ue^{\pm\tau} + \tfrac{\lambda}{R}u^{\frac{1}{2R}}e^{\pm\tau(1-1/R)}, \qquad u^{2} + \tfrac{R-1}{R^{3}}\lambda^{2}u^{\frac{2}{R}-2} = \mu$$

Sub-dominant saddles = double points of the deformed curve

For the deformed curve the degeneracy is lifted:

Double points: $x_+(\tau') = x_+(\tau'')$ and $x_-(\tau') = x_-(\tau'')$

$$\tau' = -i\theta_n, \quad \tau'' = i\theta_n, \qquad \sin(\theta_n) = \tfrac{\lambda}{R} u^{\frac{1}{2R}} \sin\left(\tfrac{1-R}{R}\,\theta_n\right),$$

Quasiclassical wave functions:

$$\Psi^E_\pm(x_\pm) \approx \frac{1}{\sqrt{\pm 2\pi \partial_\tau x_\pm}} \, e^{\mp i\phi/2} \exp\left(\mp i \int_\infty^{x_\pm} x_\mp \, dx_\pm\right)$$

$$\langle E|\mathbb{S}|E\rangle = \tfrac{1}{2\pi} \log\Lambda \quad \Rightarrow \quad e^{i\phi_{\rm pert}} \left[\log\Lambda + \sum_{n=1}^{\infty} \sqrt{\tfrac{\pi}{2}}\, \Delta_n \, e^{-\Gamma_n}\right] = e^{i\phi(-\mu)} \, \log\Lambda$$

$$\phi_{\rm pert} = \textstyle\int_{-\infty}^{\infty} x_-(\tau)\, dx_+(\tau), \qquad \Gamma_n = i \int_{-i\theta_n}^{i\theta_n} x_-(\tau)\, dx_+(\tau)$$

$$\Delta_n = \left| \left(\frac{\partial x_+}{\partial \tau}\right)_{-i\theta_n} \left(\frac{\partial x_-}{\partial \tau}\right)_{i\theta_n} \left(1 - \left(\frac{dx_-}{dx_+}\right)_{-i\theta_n} \left(\frac{dx_+}{dx_-}\right)_{i\theta_n}\right) \right|^{-1/2}$$

Subleading NPE for the free energy

$$\mathcal{F} \approx \mathcal{F}_{\rm pert} + \frac{i\sqrt{\pi}}{2\sqrt{2}\log\Lambda} \sum_{n>0} \frac{\Delta_n}{\sin\frac{\partial_\mu S_n}{2R}}\, e^{-S_n}.$$

$$A_n \sim \frac{\Delta_n}{\sin\frac{\partial_\mu S_n}{2R}} = \left(\sin^2\tfrac{\theta}{R}\left[\left(\tfrac{\partial x_+}{\partial\tau}\right)_{-i\theta_n}\left(\tfrac{\partial x_-}{\partial\tau}\right)_{i\theta_n} - \left(\tfrac{\partial x_+}{\partial\tau}\right)_{i\theta_n}\left(\tfrac{\partial x_-}{\partial\tau}\right)_{-i\theta_n}\right]\right)^{-1/2}.$$

For sine-Liouville $t_1 = t_{-1} = \lambda$:

$$A(\xi, y) = \frac{C}{\sqrt{e^{-\frac{1}{R}\chi}\sin^2\theta\,\sin^2\frac{\theta}{R}\,\left(\left(\frac{1}{R}-1\right)\cot\left(\frac{1-R}{R}\theta\right)-\cot\theta\right)}}$$

$$C = i\frac{\sqrt{\pi R}}{4\sqrt{2}\log\Lambda}$$

The pre-exponents scale as $g_s^{1/2}$ and are non-analytic in the limit $\lambda \to 0$

Bosonization in the perturbed theory

Tachyon modes = collective excitations of fermions (left and right moving waves on the Fermi surface). After second quantization these modes form a bosonic field

In (x_+, x_-) representation the fermions can be exactly bosonized.

Scnd quantized fermion fields:

$$\hat{\Psi}_\pm(x_\pm, t) = \int dE\, e^{\mp Et/2}\, \Psi^E_\pm\left(e^{\mp t} x_\pm\right)\, b(E),$$

$$\hat{\Psi}_\pm^\dagger(x_\pm, t) = \int dE\, e^{\mp Et/2}\, \overline{\Psi^E_\pm\left(e^{\mp t} x_\pm\right)}\, b^\dagger(E), \qquad \{b^\dagger(E), b(E')\} = \delta(E - E')$$

Thermal vacuum: $\langle \mu | b^\dagger(E)\, b(E') | \mu \rangle = \frac{\delta(E-E')}{1+e^{\beta(\mu+E)}}.$

Fermion phase space density operator:

$$\hat{\mathcal{U}}(x_+, x_-, t) = \frac{1}{\sqrt{2\pi}}\, e^{ix_+ x_-}\, \hat{\Psi}_-^\dagger(x_-, t)\hat{\Psi}_+(x_+, t)$$

Bosonization rules

Phase $\longrightarrow$ Bosonic field: $\hat{\Phi}_{\pm}(x_{\pm}) = V_{\pm}(x_{\pm}) + \frac{1}{2}\hat{\phi} - E\log x_{\pm} + \hat{D}_{\pm}(x_{\pm}),$

$V_{\pm}(x_{\pm}) = \sum_{n>0} t_{\pm n} x_{\pm}^{n/R},\qquad \hat{\phi} = -\frac{1}{R}\partial_E \qquad \hat{D}_{\pm}(x_{\pm}) = \sum_{n\geq 1} \frac{1}{n}\, x_{\pm}^{-n/R}\,\partial_{t_{\pm n}}.$

Fermion wave function $\longrightarrow$ Bekher-Akhieser function

$$\Psi_{\pm}^{-\mu}(x_{\pm}) = (2\pi\, x_{\pm})^{-1/2} \left\langle \mu \right| : e^{\mp i\hat{\Phi}_{\pm}(x_{\pm})} : \left| \mu \right\rangle,$$

normal product sign ($:\ :$) – all derivatives moved to the right
expectation value: $\langle\hat{\mathcal{O}}\rangle = \mathcal{Z}^{-1}\cdot\hat{\mathcal{O}}\cdot\mathcal{Z}$.
Operator formula for the fermion bilinears:

$$\hat{\Psi}_{-}^{\dagger}(x_{-})\hat{\Psi}_{+}(x_{+}) = \frac{1}{2\pi R}\,(x_{+}x_{-})^{-\frac{R+1}{2R}} : e^{-i\hat{\Phi}(x_{+},x_{-})} :,$$

where $\hat{\Phi}(x_{+},x_{-}) = \hat{\Phi}_{+}(x_{+}) + \hat{\Phi}_{-}(x_{-}).$

Pre-exponential factor as ‘annulus amplitude’

$$\left\langle : \hat{\Phi}(x_+, x_-)\hat{\Phi}(y_+, y_-) : \right\rangle_c = \log \frac{\sinh \frac{\tau(x_-)-\tau(y_+)}{2R} \sinh \frac{\tau(y_-)-\tau(x_+)}{2R}}{\sinh \frac{\tau(x_+)-\tau(y_+)}{2R} \sinh \frac{\tau(x_-)-\tau(y_-)}{2R}}$$

$$+ \log(x_+^{-1/R} - y_+^{-1/R}) + \log(x_-^{-1/R} - y_-^{-1/R}),$$

$\Rightarrow$

$$\left\langle \mu | \hat{\Psi_-}^\dagger(x_-)\hat{\Psi_+}(x_+) | \mu \right\rangle =$$

$$= \tfrac{1}{2\pi R} \frac{1}{(x_+ x_-)^{\frac{R+1}{2R}}} \exp\left[-i\Phi(x_+, x_-) - \frac{1}{2}\left\langle : \hat{\Phi}(x_+, x_-)\hat{\Phi}(x_+, x_-) : \right\rangle_c \right]$$

$$= \frac{e^{-i\Phi(x_+, x_-)}}{2\pi\sqrt{\partial_\tau x_+ \partial_\tau x_-}} \frac{1}{2R \sinh \frac{\tau(x_-)-\tau(x_+)}{2R}}.$$

NPC from bosonization

The denominator diverges on the surface of the Fermi sea, $\tau(x_+) = \tau(x_-)$, where the quasiclassics breaks down

$$2R\sin(\tfrac{1}{2R}\partial_\mu)\left\langle \mu|\hat{\Psi_-}^\dagger(x_-)\hat{\Psi_+}(x_+)|\mu\right\rangle = \frac{e^{-i\Phi(x_+,x_-)}}{2\pi\sqrt{-\partial_\tau x_+\partial_\tau x_-}} = \overline{\Psi_-^{-\mu}(x_-)}\Psi_+^{-\mu}(x_+)$$

$$N \equiv \langle\mu|\,\mathrm{Tr}\,\mathbb{I}|\mu\rangle = \int dx_+ dx_- \langle\mu|\hat{\mathcal{U}}(x_+,x_-)|\mu\rangle = -\frac{\mu}{2\pi}\log\Lambda - \frac{1}{2\pi R}\partial_\mu\mathcal{F},$$

$$\sin\left(\tfrac{1}{2R}\partial_\mu\right)N = -\frac{1}{4\pi R}\left(\log\Lambda + \partial_\mu\phi(-\mu)\right).$$

$$\Rightarrow \quad \frac{1}{\sqrt{2\pi}}\int dx_+ dx_- \frac{e^{ix_+x_- - i\Phi(x_+,x_-)}}{\sqrt{-\partial_\tau x_+\partial_\tau x_-}} = \log\Lambda.$$

The evaluation of this integral by the saddle point method gives the non-perturbative corrections to the zero mode ϕ of the bosonic field

Several fermion bilinears

$$\langle\mu|\prod_{j=1}^{n}\hat{\Psi}_{-}^{\dagger}(x_{-}^{j})\hat{\Psi}_{+}(x_{+}^{j})|\mu\rangle =$$

$$= \prod_{k=1}^{n}\frac{e^{-i\Phi(x_{+}^{k},x_{-}^{k})}}{2\pi\sqrt{\partial_\tau x_{+}^{k}\,\partial_\tau x_{-}^{k}}}\,\det_{i,j}\left(\frac{1}{2R\,\sinh\frac{\tau(x_{-}^{i})-\tau(x_{+}^{j})}{2R}}\right).$$

exact (?) after $\omega_{\pm} = e^{\tau} \to e^{i\partial_\mu}$

- The parameter τ plays the same role as the boundary parameter parametrizing the boundary cosmological constant $\mu_B = \sqrt{\mu}\cosh\tau$ in Liouville theory.
- Here there are two “boundary cosmological constants": $x_{+}(\tau)$ and $x_{-}(\tau)$.
- Two “disk amplitudes" $\Phi_{+}(x_{+}) = \int_{+}^{x} x_{-}\,dx_{+}$ and $\Phi_{-}(x_{-}) = \int_{-}^{x} x_{+}\,dx_{-}$ related by Legendre transformation

Back to (x, p) representation

1) Stationary background

$x_+(\tau), x_-(\tau) \quad \to x(\tau), w(\tau) = d\Phi/dx$
$\Phi(x)$ - the disk partition function with Dirichlet $_{\rm matter} \times$ FZZT $_{\mu_B = x}$.

$$\begin{aligned} x_+(\tau) + x_-(\tau) &= 2x(\tau), \\ x_+(\tau) - x_-(\tau) &= w(\tau + i\pi) - w(\tau - i\pi) \end{aligned}$$

The curve

$$x(\tau) = \sqrt{2\mu}\cosh\tau, \qquad w(t) = -\frac{\sqrt{2\mu}}{\pi}\tau\,\sinh\tau$$

is a $\mathbb{Z}_2$ orbifold of the universal cover of the hyperboloid $x_+x_- = \mu$, obtained by identifying the points τ and $-\tau$.

The branch points of $y = w(x)$ are images of infinitely many points $\tau = \pm i\pi n$, the (degenerate) limit of the double points of the minimal string curves

$$\tau_{mn} = i(m/b + nb), \ \ \tau_{m,-n} = i(m/b - nb)$$

Back to (x, p) representation

2) Time-dependent background

$$x(\tau) = \quad = \sqrt{2}\, e^{-\frac{1}{2R}\chi} \Big[\cosh \tau + \textstyle\sum_{k=1}^{k_{\max}} a_k \cosh\big((1-\frac{k}{R})\tau\big)\Big],$$
$$p(\tau) = \quad = \sqrt{2}\, e^{-\frac{1}{2R}\chi} \Big[\sinh \tau + \textstyle\sum_{k=1}^{k_{\max}} a_k \sinh\big((1-\frac{k}{R})\tau\big)\Big].$$

$$p(\tau) = \frac{w(\tau + i\pi_1(\tau)) - w(\tau - i\pi_1(\tau))}{2i},$$

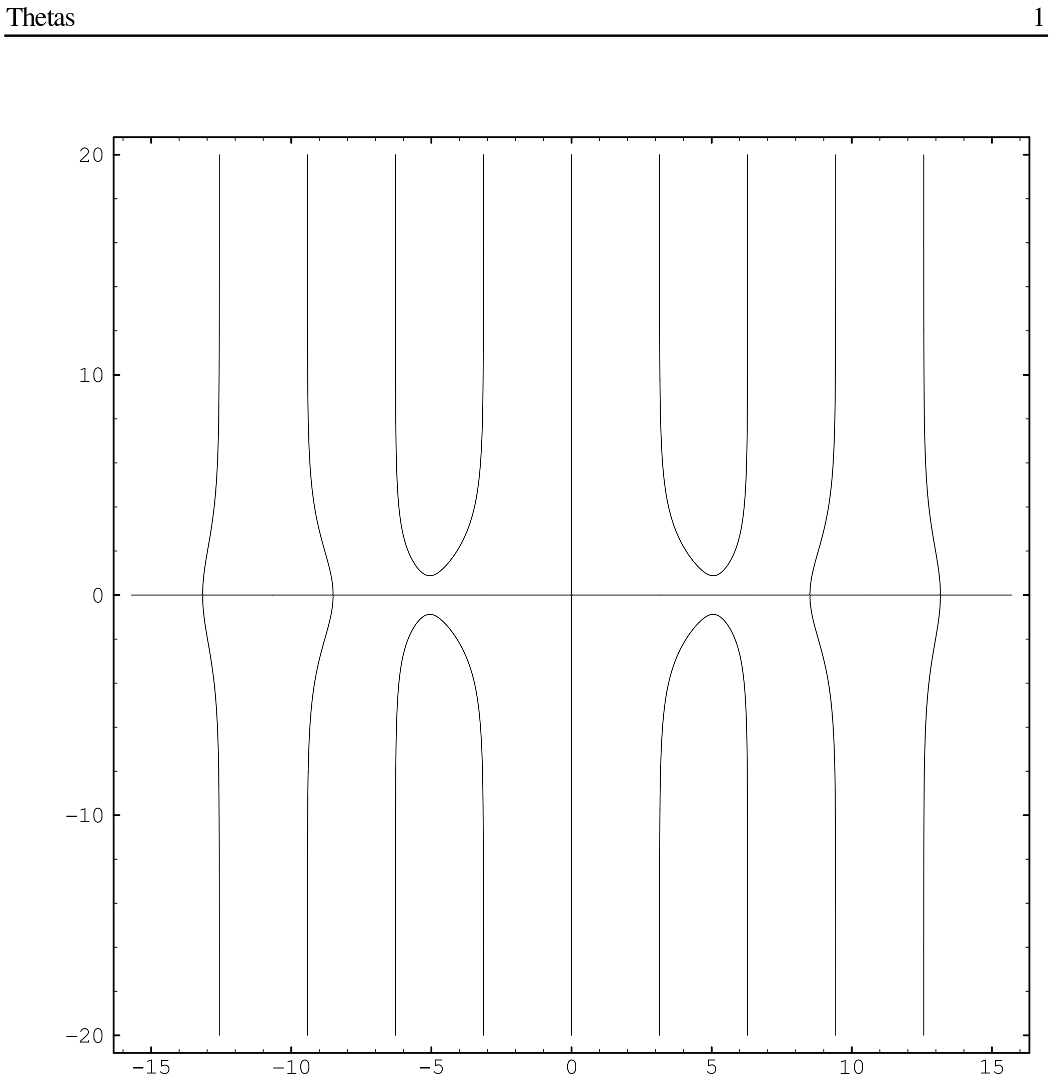
where the function $\pi_1(t)$ is obtained as the solution of the transcendental equation

$$\sin \pi_1 + \sum_{k=1}^{k_{\max}} a_k \frac{\sinh\big((1-\frac{k}{R})t\big)}{\sinh t} \sin\big((1-\frac{k}{R})\pi_1\big) = 0, \quad t \in \mathbb{R}.$$

As a consequence, the parameter τ does not uniformize the curve $y = w(x)$. In general, $w(\tau)$ will have infinitely many branch points.

Sheets of $W(x)$

horizontal: $\mathrm{Im}\,\tau$
vertical : $t = \mathrm{Re}\,\tau$
The lines give
the solutions of the
transcendental eqn
for $\pi_k(t)$



Conclusion

- The time-dependent backgrounds of the $c = 1$ string theory are described by complex curves with two singular points.

- All NPC are associated with "double points" (=vanishing cycles) of the complex curve.

- The leading exponents and the subleading factors of the non-perturbative corrections are expressed in terms of the one- and two-point functions of a bosonic field defined on the complex curve.

- The subleading correction is non-analytic in the perturbation couplings

Questions:

? Is there a D-brane interpretation of the correlation functions of the fermion bilinears?

?? If so, identify the CFT boundary conditions for these 'chiral branes'. For ϕ large and negative, these branes are extended in $\chi + \phi$ direction and localized in $\chi - \phi$ direction, or vice versa.

$$\langle \Phi(x_+, x_-) \rangle = \left\langle \sum_{n\geq 1} \left[\mathcal{T}_{n/R} x_+^{n/R} + \mathcal{T}_{-n/R} x_-^{n/R} \right] \right\rangle_{\text{sphere}} = \langle\ ???\ \rangle_{\text{disk}}$$