Fair Triangle Mesh Generation with Discrete Elastica

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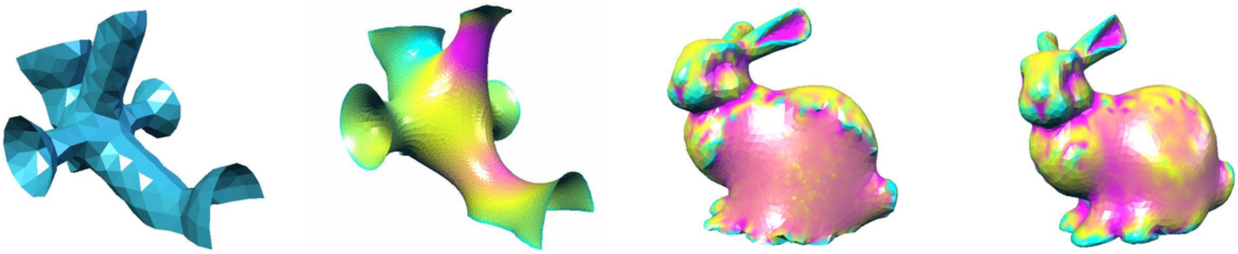

(a) (b) (c) (d)

Figure 1. Generating fair triangle meshes with discrete elastica. (a) An initial mesh outlined a complex tubular object. (b) A discrete elastica surface (mesh) obtained from the initial mesh. (c) The Stanford bunny model with a large part of the mesh removed and then triangulated. (d) The modified part of the bunny is restored as a discrete elastica. Coloring by the mean curvature is used to demonstrate a high quality of the generated meshes.

Abstract

Surface fairing, generating free-form surfaces satisfying aesthetic requirements, is important for many computer graphics and geometric modeling applications. A common approach for fair surface design consists of minimization of fairness measures penalizing large curvature values and curvature oscillations. The paper develops a numerical approach for fair surface modeling via curvature-driven evolutions of triangle meshes. Consider a smooth surface each point of which moves in the normal direction with speed equal to a function of curvature and curvature derivatives. Chosen the speed function properly, the evolving surface converges to a desired shape minimizing a given fairness measure. Smooth surface evolutions are approximated by evolutions of triangle meshes. A tangent speed component is used to improve the quality of the evolving mesh and to increase computational stability. Contributions of the paper include also an improved method for estimating the mean curvature.

KEYWORDS: *mesh fairing, elastica surfaces, discrete surface flow, Laplace-Beltrami operator.*

1 Introduction

Variational shape fairing, generating shapes satisfying certain aesthetic requirements, via minimization of fairness measures penalizing large curvature values and curvature oscillations is an active research area [10, 5, 17, 18, 19, 14, 12, 13]. A popular surface fairing measure used in various computer graphics and geometric modeling applications is the so-called *total curvature* functional [6, 5, 18]

$$\iint (k_{\max}^2 + k_{\min}^2)\, dA \qquad (1.1)$$

Here $k_{\max}$ and $k_{\min}$ are the surface principal curvatures, and dA is the surface area element. The total curvature (1.1) approximates the elastic bending energy of a thin plate [6].

Let us call the surfaces minimizing (1.1)

$$\iint (k_{\max}^2 + k_{\min}^2)\, dA \to \min \tag{1.2}$$

elastica surfaces because they generalize the famous Euler's elastica curves [3] (see also [1] for a good literature review and for a very effective method to approximate the elastica curves by polygonal curves).

The Euler-Lagrange equation corresponding to (1.2) is given by

$$\Delta_B H + 2H(H^2 - K) = 0, \tag{1.3}$$

where H and K are the mean and Gaussian curvatures, respectively, and Δ_B is the Laplace-Beltrami operator. See [4][pages 82-85] for a derivation of (1.3).

In this paper, we develop an approach for approximating elastica surfaces by triangle meshes. Our approach to minimize the total curvature functional (1.1) can be considered as a combination of the steepest descent method for (1.2) with finite differencing (approximating a smooth surface by a triangle mesh). A preliminary version of the approach was developed in [21].

Consider a family of smooth surfaces $\boldsymbol{S}(t, u, v)$, where u, v parameterize the surface and t parameterizes the family. We suppose t to be independent of u, v. Let us assume that the family evolves according to the following evolution equation

$$\frac{\partial \boldsymbol{S}(t, u, v)}{\partial t} = F\boldsymbol{N}, \qquad \boldsymbol{S}(0, u, v) = \boldsymbol{S}^{(0)}(u, v), \tag{1.4}$$

where $\boldsymbol{N}(t, u, v)$ is the unit normal vector for $\boldsymbol{S}(t, u, v)$, F is a speed function. The family parameter t can be considered as the time duration of the evolution. The gradient-descent flow for (1.2) is given by (1.4) with

$$F \equiv -\Delta_B H - 2H(H^2 - K). \tag{1.5}$$

If a surface evolved by (1.4), (1.5) converges to a limit surface $\boldsymbol{S}(\infty, u, v)$, as $t \to \infty$, then it is an elastica since the Euler-Lagrange equation (1.3) is satisfied for that limit surface.

We approximate the evolution (1.4), (1.5) by a discrete evolution of triangle meshes and use discrete analogues of the Laplace-Beltrami operator and Gaussian and mean curvatures.

One of the important contributions of the paper consists of adding to a discrete version of (1.4) a special tangent speed component used to improve the quality of the evolving mesh and to increase computational stability. The paper presents also an improved method for estimating the mean curvature of a surface approximated by a triangle mesh.

Fig. 1 illustrates how our method can be used in various geometric modeling applications. The two left images demonstrate an initial triangle mesh approximating a tubular object and a discrete elastica obtained from that initial mesh by a discrete approximation of (1.4), (1.5). The two right images show how a large missed part of a complex mesh (Stanford bunny) can be restored by a discrete elastica surface. Coloring by the mean curvature demonstrates a high quality of the generated meshes.

2 Numerical Implementation

To solve (1.4) numerically, we first approximate the time derivative term in (1.4) by its forward difference approximation

$$\frac{\partial \boldsymbol{S}(t, u, v)}{\partial t} \approx \frac{\boldsymbol{S}(t + \tau, u, v) - \boldsymbol{S}(t, u, v)}{\tau}, \quad \tau \ll 1.$$

Thus we approximate (1.4) by a discrete evolution process

$$\boldsymbol{S}(t + \tau, u, v) = \boldsymbol{S}(t, u, v) + \tau F \boldsymbol{N}(t, u, v), \tag{2.1}$$

where the speed function F is defined by (1.5). Then the surface $\boldsymbol{S}(t, u, v)$ is approximated by a triangle mesh and discrete approximations to the Laplace-Beltrami operator, Gaussian and mean curvatures, and other geometric attributes are considered. Thus the discrete evolution of surfaces (2.1) is approximated by a mesh updating process

$$\boldsymbol{P}_i^{(k+1)} = \boldsymbol{P}_i^{(k)} + \tau^{(k)} F_i^{(k)} \boldsymbol{N}_i^{(k)}, \tag{2.2}$$

where the points $\{\boldsymbol{P}_i^{(k)}\}$ form a mesh $\boldsymbol{M}^{(k)}$ obtained after k steps of the process from an initial mesh $\boldsymbol{M}^{(0)}$ approximating $\boldsymbol{S}^{(0)}(u, v)$, $\boldsymbol{N}_i^{(k)}$ is the unit mesh normal at $\boldsymbol{P}_i^{(k)}$. Here the unit mesh normal $\boldsymbol{N}$ at vertex $\boldsymbol{P}$ is computed as the normalized weighted sum of of the normals of the incident triangles, with weights equal to the areas of the triangles.

Since (1.4), (1.5) is a fourth-order partial differential equation, (the term $\Delta_B H$ involves fourth-order surface derivatives) we choose the step-size $\tau^{(k)}$ in (2.2) proportional to the squared area of the smallest triangle of $\boldsymbol{M}^{(k)}$. More precisely, we set $\tau^{(k)} = A_k^2/150$, where A_k is the minimal triangle area among the all triangles of $\boldsymbol{M}^{(k)}$.

Tangential drift for equalization of mesh triangles. Note that (2.2) is similar to an explicit finite difference scheme for a parabolic partial differential equation and, therefore, may be unstable if step-size $\tau^{(k)}$ is not small enough in a comparison with mesh triangles. Thus we can expect that a better stability of the discrete mesh evolution process can be achieved if the mesh triangles which are close to equilateral triangles and have almost the same size.

Our mesh triangle equalization technique consists of adding a tangent speed vector to (2.2). Note that adding a tangent speed component to (1.4) affects only the surface parameterization. Therefore instead of (2.2) we consider

$$\boldsymbol{P}_i^{(k+1)} = \boldsymbol{P}_i^{(k)} + \tau^{(k)} F_i^{(k)} \boldsymbol{N}_i^{(k)} + \epsilon^{(k)} \boldsymbol{T}_i^{(k)}, \qquad (2.3)$$

where $\boldsymbol{T}_i^{(k)}$ is a vector orthogonal to $\boldsymbol{N}_i^{(k)}$ and attached at $\boldsymbol{P}_i^{(k)}$, $\epsilon^{(k)}$ is a small positive parameter.

At an inner mesh vertex $\boldsymbol{P}$ let us consider the so-called umbrella-operator [16, 8] defined by

$$\mathcal{U}(\boldsymbol{P}) = \sum_i w_i \overrightarrow{\boldsymbol{PQ}_i}, \qquad (2.4)$$

where summation is taken over all neighbors of $\boldsymbol{P}$, w_i are positive weights. The geometric idea behind the umbrella operator is illustrated in Fig. 2.

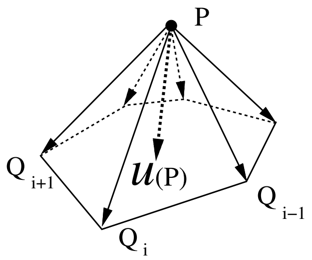

Figure 2. Umbrella operator associated with a mesh vertex *P* is defined as a weighted average of the neighbor vectors, see (2.4).

In [11] it was proposed to use the tangent component of $\mathcal{U}_0$, the umbrella operator with equal weights, for mesh regularization . The tangent component of the bi-umbrella operator $\mathcal{U}_0^2 = \mathcal{U}_0 \circ \mathcal{U}_0$ was used in [20] for similar purposes.

Following [22, 21] we use the tangent component of an area weighted bi-umbrella operator $\mathcal{U}_{\text{area}}^2$:

$$\boldsymbol{T} = -\left[\mathcal{U}_{\text{area}}^2 - (\mathcal{U}_{\text{area}}^2 \cdot \boldsymbol{N})\boldsymbol{N}\right], \qquad (2.5)$$

where

$$\mathcal{U}_{\text{area}}(\boldsymbol{P}) = \frac{1}{2An} \sum_{i=1}^{n} a_i \left(\frac{\overrightarrow{\boldsymbol{PQ}_i}}{|\overrightarrow{\boldsymbol{PQ}_i}|} + \frac{\overrightarrow{\boldsymbol{PQ}_{i+1}}}{|\overrightarrow{\boldsymbol{PQ}_{i+1}}|} \right),$$

where a_i is the area of the triangle $\boldsymbol{Q}_i\boldsymbol{P}\boldsymbol{Q}_{i+1}$, n is the number of neighboring vertices for $\boldsymbol{P}$, $A = \sum_{i=1}^{n} a_i$ is the total area of the triangles adjacent to $\boldsymbol{P}$.

If $\boldsymbol{P}$ is a boundary vertex, we set $\mathcal{U}_{\text{area}}(\boldsymbol{P}) = 0$.

According to our numerical experiments, setting $\epsilon^{(k)} = 12A_k$ produces good results. Here A_k be the minimal triangle area among the triangles of the evolving mesh $\boldsymbol{M}^{(k)}$.

Fig. 3 demonstrates equalizing mesh triangles by (2.3) with the tangent component defined by (2.5). and $\tau^{(k)} = 0$. Notice how well the proposed procedure of mesh equalization preserves the shape approximated by the original mesh.

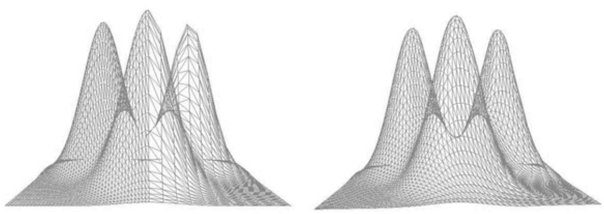

Figure 3. Left: a mesh consisted of two parts with different sampling rates. Right: tangential mesh evolution (2.3) with $\tau^{(k)} = 0$, (2.5) was used to equalize the mesh triangles.

The mesh boundary vertices are treated in a similar but more complex way since they are allowed to move along the boundary of $\boldsymbol{S}(u, v)$ only. For implementation details see [21].

Approximation of Laplace-Beltrami operator and curvatures. Recently a very efficient approximation of the Laplace-Beltrami operator for a surface approximated by a triangle mesh was proposed in [9]. The Laplace-Beltrami operator $\Delta_B(\boldsymbol{P})$ at a mesh vertex $\boldsymbol{P}$ is defined by

$$\Delta_B(\boldsymbol{P}) = \frac{3}{A} \sum_{i=1}^{n} (\cot\ \alpha_i + \cot\ \beta_i)(\boldsymbol{Q}_i - \boldsymbol{P}), \qquad (2.6)$$

where A is the total area of the triangles adjacent to $\boldsymbol{P}$, α_i and β_i are the angles $\angle \boldsymbol{P}\boldsymbol{Q}_{i-1}\boldsymbol{Q}_i$ and $\angle \boldsymbol{P}\boldsymbol{Q}_{i+1}\boldsymbol{Q}_i$, respectively.

Given a smooth surface $\boldsymbol{S}$ and a triangle mesh $\boldsymbol{M}$ approximating the surface, we use a standard angle-deficit approximation for the Gaussian curvature

$$K = \frac{3}{A}(2\pi - \sum_{i=1}^{M} \varphi_i),$$

where φ_i is the angle between $\boldsymbol{PQ}_i$ and $\boldsymbol{PQ}_{i+1}$.

Since for a smooth surface $\Delta_B \boldsymbol{S} = 2H\boldsymbol{N}$ [15], a discrete approximation of the mean curvature H can be derived from the above discrete approximation of the Laplace-Beltrami operator

$$H = \frac{1}{2}\boldsymbol{N} \cdot \Delta_B \boldsymbol{P}.$$

This approximation works very well in many applications [2, 9].

Although $H^2 - K$ is always positive for a smooth surface, it is not necessary true for discrete approximations of the Gaussian and mean curvatures. A standard approach to cope with this problem is to detect the mesh vertices where a discrete approximation of $H^2 - K$ is negative and set it equal to zero at those vertices.

However this approach is not acceptable to us since the term $H^2 - K$ is presented in (1.5) and it is not desired to have it discontinuous.

Let D denote the set of those mesh vertices for which $H \neq 0$ and $H^2 - K < 0$. We first compute

$$\lambda = \min_D \sqrt{\frac{H^2}{K}}.$$

Then we re-scale the mean curvature $H \to H/\lambda$ for the all vertices of D.

Since the quality of the mesh is improved during the evolution (2.3), $\lambda \to 1$ as $k \to \infty$.

Subdivision. In order to accelerate the mesh evolution process (2.3) we start from a coarse mesh and perform the linear one-to-four mesh subdivision when (2.3) is close to its steady-state. Fig. 4 show various stages of approximating an elastica surface via combining (2.3) with subdivision.

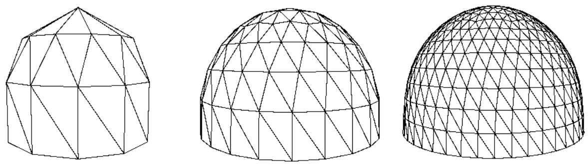

Figure 4. Starting from a coarse mesh evolved by (2.3), linear one-to-four mesh subdivision is used when (2.3) is close to its steady-state.

3 Numerical Experiments

Mesh fairing. We compare the discrete elastica flow (2.3), (1.5) with the bilaplacian flow

$$\boldsymbol{P}_i^{(k+1)} = \boldsymbol{P}_i^{(k)} - \tau \mathcal{U}_0^2(\boldsymbol{P}_i^{(k)}),$$

and a mesh evolution (2.3) by the Laplacian of mean curvature flow with speed F equal to

$$F = -\Delta_B H \tag{3.1}$$

(various numerical approaches to the Laplacian of mean curvature flow were developed in [14, 12]).

Figures 5, 6 and 7 demonstrate various stages of mesh fairing by the bilaplacian flow, the Laplacian of mean curvature flow, and the discrete elastica flow, respectively. The mesh shown in Fig. 1 (a) is used as the initial mesh. The fairing processes are also combined with subdivision. These figures and Fig. 8 demonstrate the superiority of the discrete elastica flow (2.3), (1.5) over the bilaplacian flow and the Laplacian of mean curvature flow. Coloring by the mean curvature is used to visualize the geometric quality of the meshes.

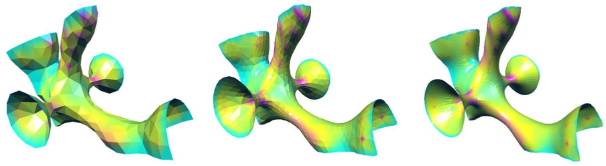

Figure 5. Mesh fairing by bilaplacian flow.

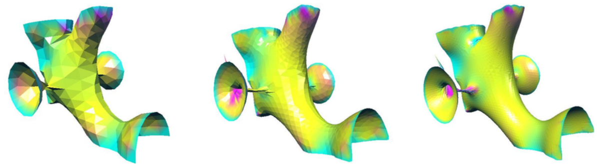

Figure 6. Mesh fairing by the Laplacian of mean curvature flow.

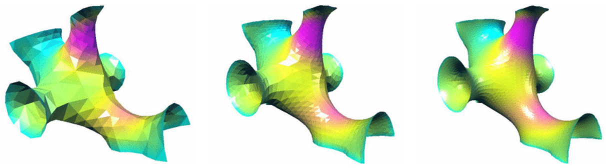

Figure 7. Mesh fairing by discrete elastica flow.

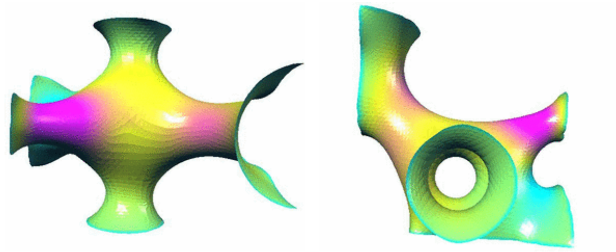

Figure 8. Discrete elastica flow produces high quality shapes.

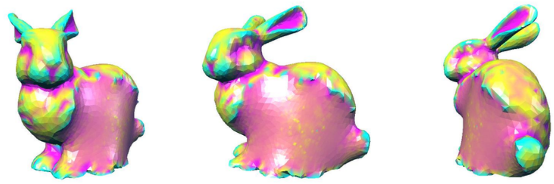

Figure 9. Stanford bunny with a large part of its flank removed and then triangulated.

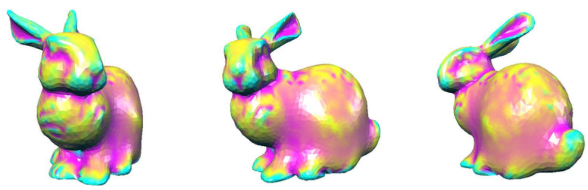

Figure 10. The bunny flank is restored by the discrete elastica flow.

Shape restoration via elastica flow. When a real-world object is digitized by a range finder, a part of shape information may be lost because of specular reflection effects, object self-occlusion, etc. The elastica flow can be used to restore missed shape parts.

Fig. 9 demonstrates the Stanford bunny having a large part of its flank removed and then triangulated. The discrete elastica flow is applied to the triangles filled the hole. The result is presented in Fig. 10. Notice a high quality of the restored part of the bunny.

4 Conclusion

The paper developed a numerical approach for generating high quality, nice-looking shapes via the discrete elastica flow. Contributions of the paper include adding a tangential speed component to the elastica gradient-descent flow for increasing computational stability of the flow, presenting an improved method for estimating the mean curvature of a surface approximated by a triangle mesh, and combining the mesh evolution approach with mesh refinement. Applications of the proposed numerical approach to mesh fairing and shape restoration were demonstrated.

Combining the developed approach with the automatic dynamic connectivity method [7] and using implicit numerical schemes for the elastica gradient-descent flow (1.4, (1.5) constitute themes for future research.

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